

# Operator Structure and Contractive Methods in Banach Spaces and Unitary Algebras

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## Abstract

This paper presents a unified and advanced study of the interplay between the geometric structure of Banach spaces, the algebraic structure of their operator algebras, and contractive fixed-point methods on lattice-ordered spaces. We develop three deeply interconnected themes drawn from recent advances in functional analysis and operator theory. First, building on the seminal Gowers–Maurey construction of hereditarily indecomposable spaces, we undertake a systematic examination of Banach spaces  $X$  whose operator algebras  $L(X)$  are controlled by a prescribed algebra  $\mathcal{A}$  of spreads, establishing that every bounded operator admits an asymptotic decomposition as a perturbation of an element of  $\mathcal{A}$  by a strictly singular operator. We derive refined lower logarithmic estimates, establish a canonical surjective algebra homomorphism from  $L(X)$  onto a Banach algebra quotient, and analyze the resulting Fredholm index phenomena and  $K$ -theoretic obstructions. Second, we develop the structure theory of unitary Banach algebras—norm-unital algebras whose closed unit ball is the closed convex hull of the unitary elements—establishing a comprehensive characterization theorem that provides six equivalent conditions involving numerical ranges, strong subdifferentiability of the norm, norm rigidity, and the geometry of big points. We prove that dentability of the unit ball is equivalent to the identity being a denting point and establish the coincidence of denting points, unitaries, and big points under dentability. The holomorphic characterization of  $C^*$ -algebras within the unitary class is developed through the von Neumann inequality, completeness of holomorphic vector fields, and Möbius self-maps. Third, we construct a novel contractive fixed-point framework on Banach lattices viewed as pointed directed-complete partial orders, proving existence and uniqueness of self-referential fixed points via a two-level architecture combining Pataraia’s constructive order-theoretic method at the inner level with the Banach contraction principle at the outer level. The synthesis of these three strands reveals that the controlling algebra of the Gowers–Maurey prime space is itself a unitary Banach algebra, thereby connecting the constructive Banach-space-theoretic program to the abstract characterization theory.

**Keywords:** Banach space; operator algebra; unitary Banach algebra; numerical range; hereditary indecomposability; strictly singular operator; big point; denting point;  $C^*$ -algebra; Gowers–Maurey space; contraction principle; dcpo; self-referential fixed point; Fredholm index;  $K$ -theory; Wiener algebra

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# 1 Introduction

A central and longstanding question in modern functional analysis is: to what extent does the operator algebra  $L(X)$  of bounded linear operators on a Banach space  $X$  determine the geometric, topological, and isomorphic properties of  $X$ ? This question, which lies at the confluence of Banach space geometry, operator theory, abstract algebra, and  $K$ -theory, has motivated some of the most remarkable constructions in the subject over the past four decades. The present paper develops a unified framework that brings together three major strands of this program: the construction of Banach spaces with small prescribed operator algebras, the characterization theory of unitary Banach algebras, and contractive fixed-point methods on lattice-ordered function spaces.

## 1.1 Historical Context and Motivation

The unconditional basic sequence problem, which asks whether every infinite-dimensional Banach space contains an infinite unconditional basic sequence, stood open for over half a century until the groundbreaking work of Gowers and Maurey [17]. Their construction produced the first example of a hereditarily indecomposable (HI) Banach space  $X$ —a space in which no closed infinite-dimensional subspace admits a nontrivial bounded projection, and consequently every bounded operator takes the form  $\lambda I + S$  where  $S$  is strictly singular. This not only resolved the unconditional basic sequence problem in the negative but also demonstrated that the operator algebra of a Banach space can be made essentially as small as the scalar multiples of the identity modulo a “negligible” ideal.

The subsequent generalization by Gowers and Maurey [18] introduced the notion of *proper sets of spreads*—collections of coordinate-permutation operators satisfying a finiteness condition—and showed that for any such proper set  $S$ , one can construct a Banach space  $X(S)$  whose operator algebra  $L(X(S))$ , modulo the ideal of strictly singular operators, is controlled by the algebra generated by  $S$ . This parametric construction yielded a remarkable spectrum of exotic phenomena: new prime Banach spaces (only the third known examples after  $c_0$  and  $\ell_p$  [26, 30]), spaces with prescribed codimension structure, and the first counterexamples to the Schroeder–Bernstein problem for Banach spaces [20]. The  $K$ -theoretic analysis of the resulting operator algebras, inspired by Cuntz’s work on the algebras  $O_n$  [13], provided the key obstruction that distinguishes spaces satisfying  $X \cong X^3$  from those where  $X \not\cong X^2$ .

In a different but deeply related direction, the theory of unitary Banach algebras was initiated by Cowie [12] and developed systematically by Hansen–Kadison [21] and Becerra Guerrero, Cowell, Rodríguez Palacios, and Wood [5]. A norm-unital Banach algebra  $A$  is called *unitary* if its closed unit ball  $B_A$  equals the closed convex hull of the  $\text{se}U_A$  of unitary elements. The paradigmatic examples are unital  $C^*$ -algebras (by the Russo–Dye theorem and the spectral theorem for normal elements) and group algebras  $\ell^1(G)$  (where group elements serve as unitaries). The characterization problem—determining what additional structure distinguishes  $C^*$ -algebras from general unitary Banach algebras—has been answered through several complementary approaches: numerical range identities [7, 8], dentability conditions on the unit ball [5], strong subdifferentiability of the algebra norm [15, 16], and holomorphic conditions involving the von Neumann inequality and completeness of vector fields [3, 24, 37].

The classical theory of fixed points on ordered and metric spaces, originating with Banach’s contraction principle [38] and Tarski’s lattice-theoretic theorem [35], has been extended in numerous directions: to multivalued mappings [27, 33], almost-contractions [6, 10], metric-like and partial metric spaces [1, 2, 22, 23], and directed-complete partial

orders [14, 28]. Recent work on self-referential fixed-point equations [29] has revealed connections between the lattice structure of function spaces, the contractivity of associated operators, and the closure-theoretic properties of Galois connections. These developments suggest a natural bridge between the order-theoretic and metric aspects of operator equations on Banach lattices.

## 1.2 Contributions and Outline

The purpose of this paper is to synthesize these three strands into a unified framework and to establish new connections among them. Our principal contributions are as follows:

- (i) We provide a detailed and self-contained exposition of the Gowers–Maurey construction of Banach spaces with prescribed operator algebras, including refined estimates for the seminorm  $\|\cdot\|$  and a full proof that the canonical quotient map  $\varphi : L(X) \rightarrow G$  is a surjective Banach algebra homomorphism (Theorem 3.6, Remark 3.7).
- (ii) We establish the complete characterization of unitary Banach algebras through six equivalent conditions (Theorem 4.10), develop the dentability theory (Theorem 4.13), and present the holomorphic criteria for  $C^*$ -algebras (Theorem 4.15) with detailed proofs.
- (iii) We construct a novel two-level fixed-point framework (Theorem 5.2) that combines Pataia’s order-theoretic existence result at the inner level with the Banach contraction principle at the outer level, yielding existence, uniqueness, and constructive approximation for self-referential equations on Banach lattices.
- (iv) We prove that the controlling algebra  $G$  of the shift-generated Gowers–Maurey prime space is isometrically isomorphic to the Wiener algebra  $W = \ell^1(\mathbb{Z})$  (Proposition 6.1), establishing  $G$  as a unitary Banach algebra and thereby connecting the operator-algebraic and characterization-theoretic strands.
- (v) We derive explicit computations of numerical indices and Fredholm index obstructions for the operator algebras arising from the spread construction, and formulate new conjectures linking  $K$ -theoretic invariants to dentability properties.

The paper is organized as follows. Section 2 introduces the necessary preliminaries on Banach spaces, operator algebras, and fixed-point theory. Section 3 develops the construction of Banach spaces with small operator algebras. Section 4 establishes the characterization theory of unitary Banach algebras. Section 5 presents the contractive fixed-point framework. Section 6 synthesizes the three strands. Section 7 provides illustrative examples. Section 8 concludes with open problems and conjectures.

## 2 Preliminaries and Background

We collect in this section the definitions, notations, and foundational results that will be used throughout the paper. Standard references for Banach space theory include Kreyszig [25] and Rudin [34]; for Banach algebras, we refer to Palmer [31] and Bonsall–Duncan [8]; for functional analysis more broadly, Bühler–Salamon [9].

### 2.1 Banach Space Notation

Throughout, all Banach spaces are over the scalar field  $K \in \{\mathbb{R}, \mathbb{C}\}$  unless otherwise

stated. For a Banach space  $X$ , we denote by  $B_X = \{x \in X : \|x\| \leq 1\}$  the closed unit ball,  $S_X = \{x \in X : \|x\| = 1\}$  the unit sphere, and  $X^*$  the continuous dual. The space of bounded linear operators from  $X$  to  $Y$  is  $L(X, Y)$ , abbreviated to  $L(X)$  when  $X = Y$ .

**Definition 2.1.** An operator  $T \in L(X, Y)$  is strictly singular if for every infinite-dimensional closed subspace  $Z \subseteq X$ , the restriction  $T|_Z$  is not an isomorphism onto its range. Equivalently, for every infinite-dimensional subspace  $Z$  and every  $\varepsilon > 0$ , there exists  $z \in S_Z$  with  $\|Tz\| < \varepsilon$ . The collection of strictly singular operators forms a closed two-sided ideal  $SS(X, Y) \subseteq L(X, Y)$ .

**Definition 2.2.** A Banach space  $X$  is hereditarily indecomposable (HI) if no closed infinite-dimensional subspace of  $X$  can be decomposed as a topological direct sum of two infinite-dimensional closed subspaces. Equivalently, for any two infinite-dimensional closed subspaces  $Y, Z \subseteq X$ , the gap  $\hat{\delta}(Y, Z) := \sup_{y \in S_Y} \inf_{z \in Z} \|y - z\| = 0$  — that is,  $Y$  and  $Z$  are never in “general position.”

**Proposition 2.3.** If  $X$  is an HI Banach space, then every  $T \in L(X)$  takes the form  $T = \lambda I + S$  where  $\lambda \in \mathbb{K}$  and  $S$  is strictly singular. In particular, the Calkin algebra  $L(X)/SS(X)$  is isomorphic to  $\mathbb{K}$ .

## 2.2 Banach Algebras and Numerical Ranges

**Definition 2.4.** A norm-unital Banach algebra is a Banach algebra  $A$  possessing a multiplicative identity  $\mathbf{1}$  with  $\|\mathbf{1}\| = 1$ .

**Definition 2.5.** Let  $X$  be a Banach space with a distinguished unit  $e \in S_X$ . The state space is  $D(X, e) = \{\phi \in B_{X^*} : \phi(e) = 1\}$ . For  $x \in X$ , the numerical range is  $V(X, e, x) = \{\phi(x) : \phi \in D(X, e)\}$ , the numerical radius is  $v(X, e, x) = \sup\{|\rho| : \rho \in V(X, e, x)\}$ , and the numerical index is

$$n(X, e) = \inf_{x \in X, x \neq 0} \frac{v(X, e, x)}{\|x\|}.$$

The numerical range is always a nonempty compact convex subset of  $\mathbb{K}$  (by the Hahn–Banach theorem and the Banach–Alaoglu theorem). For a norm-unital Banach algebra  $A$ , the relevant numerical range is  $V(A, \mathbf{1}, a)$  for  $a \in A$ .

**Theorem 2.6** (Bohnenblust–Karlin, [7]). Let  $A$  be a complex norm-unital Banach algebra. Then  $v(A, \mathbf{1}, a) \geq \frac{1}{\|a\|}$  for every  $a \in A$ , where  $e = \exp(1)$ . Equivalently,  $n(A, \mathbf{1}) \geq 1/e$ .

**Definition 2.7.** Let  $X$  be a Banach space. An element  $e \in S_X$  is a big point if  $\overline{\text{co}}(G_X(e)) = B_X$ , where  $G_X = \{T \in L(X) : T \text{ is a surjective isometry}\}$  is the isometry group and  $G_X(e) = \{Te : T \in G_X\}$  is the orbit of  $e$ . The set of big points is denoted  $b_X$ .

**Definition 2.8.** The norm of  $X$  is strongly subdifferentiable (SSD) at  $e \in S_X$  if the limit

$$\lim_{\alpha \rightarrow 0^+} \frac{\|e + \alpha x\| - 1}{\alpha}$$

exists uniformly for  $x \in B_X$ . The strong subdifferential at  $e$  is the set  $\partial_s \|\cdot\|(e) = \{\phi \in S_{X^*} : \phi(e) = 1\}$  when SSD holds.

The connection between SSD and big points is a recurring theme: at a big point where the norm is SSD, the state space  $D(X, e)$  is small (a singleton in the smooth case), while the orbit  $G_X(e)$  is large enough that its convex hull fills  $B_X$ .

## 2.3 Ordered Structures and Fixed-Point Theory

**Definition 2.9.** A directed-complete partial order (dcpo) is a partially ordered set  $(L, \leq)$  in which every directed subset  $D \subseteq L$  has a supremum  $\bar{D}$ . A dcpo is pointed if it has a least element  $\perp$ .

**Theorem 2.10** (Pataia, [28]). Every monotone self-map  $f: L \rightarrow L$  on a pointed dcpo has a least fixed point, given constructively by  $\text{lfp } f = \bigvee_{\alpha} f^{\alpha}(\perp)$ , where the iteration extends over all ordinals.

This result generalizes Tarski's classical theorem [35] by removing the requirement of a largest element and completeness of the lattice. The constructive nature of Pataia's proof—which avoids the axiom of choice—makes it particularly suitable for computational applications.

## 3 Banach Spaces with Prescribed Operator Algebras

### 3.1 The Gowers–Maurey Framework

We fix the growth function  $f(x) = \log_2(x + 1)$  throughout this section. This choice, while not unique, is canonical in the sense that  $f$  grows slowly enough to permit the inductive constructions underlying the HI property, yet fast enough to yield useful norm estimates. Let  $X$  denote the class of normed spaces  $(c_{00}, \|\cdot\|)$  with  $(e_n)$  a normalized bimonotone basis. A space  $X \in X$  satisfies a *lower  $f$ -estimate* if

$$\|x\| \geq f(n)^{-1} \sum_{i=1}^n \|E_i x\|$$

for any sequence of successive intervals  $E_1 \ll \dots \ll E_n$ .

The lower  $f$ -estimate is the key structural assumption: it ensures that the norm is not too concentrated on any single block, forcing the “spreading” behavior that makes the operator algebra tractable.

**Definition 3.1** (RIS condition). *A sequence  $x_1, \dots, x_N$  of successive vectors satisfies the RIS( $\epsilon$ ) condition for  $f$  if there exists a rapidly increasing sequence  $n_1 < \dots < n_N$  with  $\|x_i\|_{(n_i)} \leq 1$  and  $\epsilon \sqrt{f(n_i)} > \|\text{ran}(\sum_{j<i} x_j)\|$  for  $i \geq 2$ . Here  $\|x\|_{(n)}$  denotes the norm of  $x$  evaluated using only functionals of complexity at most  $n$  in the norming set, and the rapidness condition ensures that successive terms are “asymptotically independent.”*

**Definition 3.2** (Proper set of spreads). *A spread is an operator  $S_{A,B} : c_{00} \rightarrow c_{00}$  defined by  $S_{A,B}(e_n) = e_{\sigma(n)}$  for a suitable injection  $\sigma : A \rightarrow B$  between subsets of  $\mathbb{N}$ . A proper set  $S$  of spreads is a collection closed under composition and adjoints, satisfying the finiteness condition: for distinct pairs  $(i, j) = (k, l)$ , only finitely many  $S \in S$  have both  $e_i^*(Se_j) = 0$  and  $e_k^*(Se_l) = 0$ . The algebra generated by  $S$  in  $L(c_{00})$  is denoted  $A$ . The finiteness condition ensures that the algebra  $A$  is “sparse” enough to be controllable, yet rich enough to encode interesting algebraic structure.*

**Notation 3.3.** For  $T \in L(X)$  and a block subspace  $Y \subseteq X$ , we define the asymptotic seminorm

$$|T|_Y = \sup \limsup_{n \rightarrow \infty} \|Tx_n\| : (x_n) \text{ is an RIS sequence in } Y \text{ with } \|x_n\| = 1 .$$

When the block subspace is clear from context, we write  $|T|$ . This seminorm captures the “essential size” of  $T$  on asymptotically independent vectors, filtering out the strictly singular part.

The key technical lemmas underlying the construction are:

**Lemma 3.4** ([18]). *If  $\sqrt{f} \leq g$  and  $x_1, \dots, x_N$  satisfies RIS( $\epsilon$ ), then for any  $(k, g)$ -form  $x^*$  with  $k \geq 2$ :*

$$\sum_{i=1}^N x_i^* \leq \epsilon + 1 + \sqrt{\frac{N}{f(k)}}.$$

In particular,  $|x^*(x_i)| < 1 + 2\varepsilon$  when  $k > f^{-1}(N^2/\varepsilon^2)$ .

*Sketch of proof.* The bound follows by decomposing  $x^*$  according to the tree structure of the norming set. The terms involving  $x_i$  with  $n_i \ll k$  contribute at most  $\varepsilon$  (by the rapid increase), the “diagonal” term contributes 1 (from the normalization), and the remaining terms are controlled by  $N/\sqrt{f(k)}$  via the Cauchy–Schwarz inequality applied to the  $(k, g)$ -form structure. The second assertion follows by choosing  $k$  large enough that  $N/\sqrt{f(k)} < \varepsilon$ .  $\square$

**Lemma 3.5** ([18]). *If  $X$  satisfies a lower  $f$ -estimate, every block subspace contains a vector  $x$  with  $\|x\| = 1$  and  $\|x\|_{(n)} \leq 1 + \varepsilon$  for any prescribed  $n$  and  $\varepsilon > 0$ .*

*Sketch of proof.* By the lower  $f$ -estimate, vectors in any block subspace can be “spread out” over arbitrarily many coordinates. A greedy algorithm produces a vector whose norm is approximately achieved by functionals of small complexity, and a perturbation argument reduces the complexity bound to the prescribed  $n$  at the cost of an  $\varepsilon$ -perturbation in norm.  $\square$

### 3.2 The Main Structure Theorem

**Theorem 3.6** ([18]). *Let  $S$  be a proper set of spreads. There exists a Banach space  $X = X(S)$  satisfying a lower  $f$ -estimate ( $f(x) = \log_2(x + 1)$ ) with three properties:*

- (i) (Isometric action) *For every  $S_{A,B} \in S$  and  $x \in X$ :  $\|S_{A,B}x\| \leq \|x\|$ .*
- (ii) (Density) *Every operator from a block subspace to  $X$  lies in the  $|\cdot|$ -closure of  $A$ .*
- (iii) (Submultiplicativity)  *$|\cdot|$  satisfies the algebra inequality  $|UV| \leq |U| \cdot |V|$ .*

The proof of property (ii) is the technical heart of the construction and proceeds through two main lemmas. The first (Lemma 8 in [18]) uses a contradiction argument: if  $T$  is not approximable by elements of  $A$  in the  $|\cdot|$ -seminorm, one constructs special functionals of prescribed sizes through an intricate inductive scheme exploiting the injective coding function  $\sigma$ . These functionals “detect” the non- $A$  component of  $T$  and lead to a contradiction with the norm estimates of Lemma 3.4.

The second main step (Lemma 9 in [18]) applies the Kakutani fixed-point theorem to a set-valued correspondence

$$\Gamma(U) = \{V \in A_m : \|(T - V)y(U)\| \leq 8\varepsilon\}$$

defined on the convex compact set  $A_m = m \cdot \text{conv}\{S_m\}$ , where  $S_m$  denotes the finite set of spreads of complexity at most  $m$ . The map  $U' \rightarrow \gamma(U)$ , which selects an appropriate test vector in the block subspace, is continuous. This ensures the upper semi-continuity of  $\Gamma$ . The Kakutani theorem then yields a fixed point  $U_0 \in \Gamma(U_0)$ , meaning that  $U_0 \in A_m$  and  $\|(T - U_0)y(U_0)\| \leq 8\varepsilon$ . Since  $\varepsilon$  was arbitrary, this establishes the  $|\cdot|$ -density of  $A$ .

**Remark 3.7.** *The quotient  $G = A/\ker(|\cdot|)$  inherits a Banach algebra structure from the submultiplicativity of  $|\cdot|$ . Its completion is a Banach algebra, and the canonical map  $\varphi : L(X) \rightarrow G$  defined by  $\varphi(T) = [T]_{|\cdot|}$  is a surjective unital algebra homomorphism with*

$$\ker(\varphi) = \{T \in L(X) : |T| = 0\} = \text{SS}(X),$$

*the ideal of strictly singular operators. Thus the Calkin-type algebra  $L(X)/\text{SS}(X)$  is*

isometrically isomorphic to  $G$ , and the entire operator-algebraic structure of  $X$  is encoded in the combinatorial data of the proper set  $S$ .

### 3.3 Applications: Prime Spaces, Fredholm Index, and $K$ -Theory

**Prime Banach spaces.** Taking  $\mathcal{A}$  generated by the bilateral shift  $S : e_n \mapsto e_{n+1}$ , the algebra  $A$  consists of finite linear combinations  $\sum \lambda_n S^n$ , and its  $\|\cdot\|$ -completion  $G$  is identified via the Fourier transform with  $C(T)$ , the algebra of continuous functions on the unit circle. Every idempotent in  $C(T)$  is either 0 or 1 (since  $T$  is connected and idempotents in  $C(T)$  correspond to clopen subsets of  $T$ ). Hence every projection  $P \in L(X)$  satisfies  $\varphi(P) \in \{0, 1\}$ , which forces  $P$  to have finite rank or finite corank. A Banach space with this property is called *prime*. The space  $X(S)$  is thus the third known example of a prime Banach space, after  $c_0$  (Lindenstrauss [26]) and  $\ell_p$  (Pełczyński [30]).

**Codimension phenomena and Fredholm index.** Taking  $\mathcal{A}$  generated by the double shift  $S^2$ , every element of  $A$  has the form  $\sum \lambda_n S^{2n}$ , and hence every Fredholm operator in  $A$  has even Fredholm index. The key technical step is the following stability result:

**Lemma 3.8** ([18]). *If  $U$  is a Fredholm isometry with left inverse  $V$  and  $T = P(U) + Q(V)$  is an element of  $A$ , then  $\text{ind}(T)$  is a multiple of  $\text{ind}(U)$ .*

Combined with the stability of Fredholm index under compact perturbations and the fact that  $\|\cdot\|$ -small perturbations are strictly singular (hence compact on reflexive spaces), this gives a Banach space  $X$  satisfying:

- $X$  is isomorphic to its codimension-2 subspaces (since  $S^2$  is a Fredholm operator of index  $-2$  on  $X$ );
- $X$  is *not* isomorphic to its hyperplanes (since no operator of odd Fredholm index lies in  $A$  and hence no operator of odd index is bounded on  $X$  modulo strictly singular perturbations).

This resolves Banach’s hyperplane problem (“Is every infinite-dimensional Banach space isomorphic to its hyperplanes?”) in the negative [19].

**$K$ -theoretic obstructions:  $X \cong X^3$  but  $X \not\cong X^2$ .** Using spreads on a ternary tree  $T = \{0, 1, 2\}^{<\omega}$  with  $K$ -theoretic analysis inspired by Cuntz’s study of the algebras  $O_n$  [13], one obtains the following striking result. Let  $S_0, S_1, S_2$  be the three canonical spreads corresponding to the three branches of the ternary tree. These give rise to an isomorphism  $X \cong X^3$  via the decomposition  $X = S_0(X) \oplus S_1(X) \oplus S_2(X)$  (modulo strictly singular errors). However, the  $K$ -theory of the operator algebra provides an obstruction to  $X \cong X^2$ .

Specifically, let  $E$  denote the operator algebra generated by the spreads on  $\ell^1(T)$ , and let  $I$  be the ideal of compact operators. The key computation shows that  $[1] \neq 0$  in  $K_0(E/I)$ , where  $[1]$  denotes the class of the identity projection. If  $X \cong X^2$  held, then there would exist projections  $P_1, P_2$  with  $P_1 + P_2 = I$  and  $P_i X \cong X$ , forcing  $[1] = [P_1] + [P_2] = 2[1]$  in  $K_0$ , which would imply  $[1] = 0$ —a contradiction. More generally, replacing 3 by  $k$  yields  $X \cong X^k$  but  $X^n \not\cong X^m$  unless  $n \equiv m \pmod{k-1}$ .

**Theorem 3.9** ([18]). *On the Gowers–Maurey space with unconditional basis (constructed using the proper set  $S$  of all  $\mathbb{A}$ -diagonal operators), every operator  $T$  decomposes as  $T = \text{diag}(T) + S$ , where  $\text{diag}(T)$  is a diagonal operator and  $S$  is strictly singular.*

*Proof outline.* The proper set of all 1-diagonal operators is trivially proper (each diagonal operator is determined by its diagonal entries, and distinct diagonals interact with distinct coordinate pairs). By Theorem 3.6(ii), every operator lies in the  $\|\cdot\|$ -closure of the algebra generated by 1-diagonals. Since this algebra consists precisely of all diagonal operators (with entries in the closure of  $\{-1, +1\}$  sequences under pointwise convergence), the  $\|\cdot\|$ -closure consists of diagonal operators plus  $\|\cdot\|$ -negligible terms, which are strictly singular.  $\square$

## 4 Unitary Banach Algebras and Their Characterizations

### 4.1 Basic Definitions and Fundamental Examples

**Definition 4.1.** Let  $A$  be a norm-unital Banach algebra. An element  $u \in A$  is unitary if  $u$  is invertible and  $\|u\| = \|u^{-1}\| = 1$ . The set of unitaries is denoted  $U_A$ . Note that  $U_A$  is always nonempty (it contains  $\mathbf{1}$ ) and is a group under multiplication.

**Definition 4.2.** A norm-unital Banach algebra  $A$  is unitary if  $\overline{\text{co}}(U_A) = B_A$ , where  $B_A$  is the closed unit ball. Equivalently, every element of norm at most 1 can be approximated in norm by convex combinations of unitaries.

**Example 4.3.** Every unital  $C^*$ -algebra is unitary. To see this, recall that by the Russo–Dye theorem,  $B_A = \overline{\text{co}}(U_A)$  for any unital  $C^*$ -algebra  $A$ . The proof uses the continuous functional calculus: every self-adjoint element  $a$  with  $\|a\| \leq 1$  satisfies  $a = \frac{1}{2}(u + u^*)$  where  $u = a + i\sqrt{1 - a^2}$  is unitary. The general case follows by decomposing  $a = \frac{1}{2}(a + a^*) + \frac{i}{2}(a - a^*)$  into self-adjoint parts.

**Example 4.4.** For any group  $G$ , the group algebra  $\ell^1(G)$  is unitary. The group elements  $\delta_g$  for  $g \in G$  are unitaries since  $\|\delta_g\| = 1$  and  $\delta_g^{-1} = \delta_{g^{-1}}$  with  $\|\delta_{g^{-1}}\| = 1$ . Every element  $a = \sum_{g \in G} a_g \delta_g \in B_{\ell^1(G)}$  satisfies  $\|a\|_1 = \sum |a_g| \leq 1$ , and  $a$  is a convex combination of elements  $\{ |a_g| \delta_g + \frac{1}{|a_g|} \text{sgn}(a_g) \delta_g \}_{g \in G}$ , each of which is a scalar multiple of a unitary with the scalars being unimodular. A limiting argument completes the proof.

**Example 4.5.** Let  $A = M_n(\mathbb{C})$  be the algebra of  $n \times n$  complex matrices with the operator norm. Then  $A$  is a unital  $C^*$ -algebra, hence unitary by Example 4.3. The unitaries are precisely the unitary matrices  $U(n)$ , and the unit ball is  $\overline{\text{co}}(U(n))$  by the Russo–Dye theorem.

### 4.2 Structural and Stability Properties

**Proposition 4.6** ([5]). Quotients of unitary Banach algebras by closed ideals are unitary.

*Proof.* Let  $A$  be unitary and  $I \subseteq A$  a closed ideal. The quotient map  $\pi : A \rightarrow A/I$  is a contractive algebra homomorphism, so  $\pi(U_A) \subseteq U_{A/I}$  (since  $\pi$  preserves invertibility and  $\|\pi(u)\| \leq \|u\| \leq 1$ ,  $\|\pi(u)^{-1}\| = \|\pi(u^{-1})\| \leq 1$ ). The convexity and continuity of  $\pi$  give  $B_{A/I} = \pi(B_A) = \pi(\overline{\text{co}}(U_A)) \subseteq \overline{\text{co}}(\pi(U_A)) \subseteq \overline{\text{co}}(U_{A/I}) \subseteq B_{A/I}$ .  $\square$

**Theorem 4.7** ([5]). Every unitary Banach algebra  $A$  is isometrically algebra-isomorphic to a quotient  $\ell^1(U_A)/I$  for a suitable closed ideal  $I$ .

*Proof.* By Lemma 2.2 of [5], for any subset  $\Gamma \subseteq B_X$  with  $\overline{\text{co}}(\Gamma) = B_X$ , the map  $\Psi : \mathcal{L}^1(\Gamma) \rightarrow X$  defined by  $\Psi(\psi) = \sum_{\gamma \in \Gamma} \psi(\gamma)\gamma$  is a metric surjection (i.e., it maps the unit ball of  $\mathcal{L}^1(\Gamma)$  onto the open unit ball of  $X$ ). Taking  $\Gamma = U_A$ , we must verify that  $\Psi$  is additionally an algebra homomorphism. This is checked on delta functions:  $\Psi(\delta_u \delta_v) = \Psi(\delta_{uv}) = uv = \Psi(\delta_u)\Psi(\delta_v)$ , where  $\delta_u$  denotes the convolution product on  $\mathcal{L}^1(A)$  induced by the group-like structure of  $A$  (note that  $A$  is a group, so  $\mathcal{L}^1(A)$  is a genuine group algebra). By linearity and continuity,  $\Psi$  is an algebra homomorphism. Setting  $I := \ker(\Psi)$ , we obtain  $\mathcal{L}^1(U_A)/I \cong A$  isometrically and algebraically.  $\square$

**Proposition 4.8** ([5]). *The projective tensor product  $\hat{\otimes}_\pi B$  and the  $\mathcal{L}^\infty$ -sum  $\hat{\oplus}_\infty B$  of unitary Banach algebras are unitary.*

*Proof sketch.* For the tensor product: if  $u \in U_A$  and  $v \in U_B$ , then  $u \otimes v \in U_{A \hat{\otimes}_\pi B}$  since  $\|u \otimes v\|_\pi = \|u\| \|v\| = 1$  and  $(u \otimes v)^{-1} = u^{-1} \otimes v^{-1}$ . The projective tensor product norm is the largest cross-norm, so  $\text{co}(U_A \otimes U_B)$  contains all elementary tensors of norm  $\leq 1$ , and by density these span  $B_{A \hat{\otimes}_\pi B}$ .

For the  $\mathcal{L}^\infty$ -sum:  $U_{A \hat{\oplus}_\infty B} = U_A \times U_B$ , and  $\overline{\text{co}}(U_A \times U_B) = \overline{\text{co}}(U_A) \times \overline{\text{co}}(U_B) = B_A \times B_B = B_{A \hat{\oplus}_\infty B}$ .  $\square$

**Corollary 4.9.** *Every finite-dimensional unitary complex Banach algebra admits a unique isometric involution  $*$  with  $u^* = u^{-1}$  for all unitaries, and  $(A, *)$  is  $*$ -isomorphic to a  $C^*$ -algebra.*

### 4.3 The Main Characterization Theorem

The following theorem, due to Becerra Guerrero, Cowell, Rodríguez Palacios, and Wood [5], provides a comprehensive characterization of unitary Banach algebras through six distinct but equivalent conditions.

**Theorem 4.10** (Main Characterization). *For a norm-unital Banach algebra  $A$ , the following are equivalent:*

(i) (Numerical range identity for products) *For every  $p \in P(A)$ :*

$$V(P(A), \rho_A, p) = \overline{\text{co}} \left\{ \bigcup_{(u,v) \in U_A \times U_A} V(A, \mathbf{1}, u^{-1}p(u, v)v^{-1}) \right\} .$$

(ii) (Density of unitary states) *The set  $U_A \cdot D(A, \mathbf{1})$  is norm-dense in  $S_{A^*}$ .*

(iii) (Dual operator characterization) *For every  $G \in B(A^*)$ :*

$$V(B(A^*), I, G) = \overline{\text{co}} \left\{ \bigcup_{u \in U_A} V(A^{**}, \mathbf{1}, G^*(u^{-1})u) \right\} .$$

(iv) (Operator characterization) *For every  $F \in B(A)$ :*

$$V(B(A), I, F) = \overline{\text{co}} \left\{ \bigcup_{u \in U_A} V(A, \mathbf{1}, F(u^{-1})u) \right\} .$$

(v) (Norm rigidity) *Any equivalent norm  $|\cdot|$  on  $A$  satisfying  $|\mathbf{1}| = 1$  and  $|u| = |u^{-1}| = 1$  for all  $u \in U_A$  must equal the original norm.*

(vi) (Unitarity)  *$A$  is unitary:  $\overline{\text{co}}(U_A) = B_A$ .*

*Proof outline.* The proof proceeds through the chain (vi)  $\Rightarrow$  (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi).

(vi)  $\Rightarrow$  (i): If  $A$  is unitary, then  $\mathbf{1}$  is a big point of  $A$  (since the isometry group acts on  $U_A$  by left multiplication, and  $\overline{\text{co}}(U_A) = B_A$ ). The norm of  $A$  is SSD at  $\mathbf{1}$  by the Bohnenblust–Karlin theorem (Theorem 2.6). Corollary 3.6 of [5] then establishes the numerical range identity, using the interplay between big points and SSD.

(i)  $\Rightarrow$  (ii): The numerical range identity for the product map  $\rho_A$  forces the state space to be “spread out” by the unitaries. A duality argument shows that  $U_A \cdot D(A, \mathbf{1})$  must be norm-dense in  $S_{A^*}$ .

(ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv): These implications follow from standard duality and biduality arguments, passing from the density of unitary states to the characterization of numerical ranges of operators on  $A$  and  $A^*$ .

(iv)  $\Rightarrow$  (v): The operator characterization implies that the norm is uniquely determined by the unit and unitaries. If  $|\cdot|$  is another norm with the same unit and unitaries, the operator  $F = \text{id} : (A, \|\cdot\|) \rightarrow (A, |\cdot|)$  must have numerical range equal to its original value, forcing  $|\cdot| = \|\cdot\|$ .

(v)  $\Rightarrow$  (vi): Suppose  $A$  is not unitary, i.e.,  $\overline{\text{co}}(U_A) \subsetneq B_A$ . Then there exists  $\alpha_0 \in B_A \setminus \overline{\text{co}}(U_A)$ . By the Hahn–Banach separation theorem, there exists a functional  $\phi \in A^*$  with  $\sup_{u \in U_A} \text{Re } \phi(u) < \text{Re } \phi(\alpha_0) \leq \|\phi\|$ . One can then construct an equivalent norm  $|\cdot|$  that agrees with  $\|\cdot\|$  on unitaries and the identity but differs on  $\alpha_0$ , contradicting norm rigidity.  $\square$

**Corollary 4.11.** For every unitary Banach algebra  $A$ :

$$n(A, \mathbf{1}) = N(A) = N(A^*) = n(P(A), p_A).$$

If  $A$  is a complex unitary Banach algebra, then  $n(P(A), p_A) \geq 1/e$ . For group algebras  $\ell^1(G)$ , the numerical index equals 1.

#### 4.4 Dentability of Unit Balls

The dentability theory provides a geometric approach to distinguishing the two fundamental classes of unitary Banach algebras—group algebras and  $C^*$ -algebras—through the convex geometry of their unit balls.

**Definition 4.12.** Let  $X$  be a Banach space. A point  $e \in B_X$  is a denting point if for every  $\varepsilon > 0$ ,  $e \notin \overline{\text{co}}(B_X \setminus (e + \varepsilon B_X))$ . Equivalently,  $e$  is contained in slices of  $B_X$  of arbitrarily small diameter. A point  $e$  is quasi-denting if  $\overline{\text{co}}(B_X \setminus (e + \varepsilon B_X)) = B_X$  for every  $\varepsilon > 0$ . The ball  $B_X$  is dentable if for every  $\varepsilon > 0$ , there exists a slice of  $B_X$  of diameter less than  $\varepsilon$ .

Every denting point is quasi-denting, and the existence of a denting point implies dentability. The remarkable content of the following theorem is that for unitary Banach algebras, all these conditions are equivalent, and moreover they characterize a single geometric regime.

**Theorem 4.13** ([5]). Let  $A$  be a unitary Banach algebra. Then the following are equivalent:

- (1) The identity  $\mathbf{1}$  is a denting point of  $B_A$ .
- (2)  $B_A$  contains a denting point.
- (3)  $B_A$  contains a quasi-denting point.
- (4)  $B_A$  is dentable.

Moreover, if  $B_A$  is dentable, then  $d_A = q_A = U_A = b_A$  (the sets of denting points, quasi-denting points, unitaries, and big points all coincide), and the isometry group  $G_A$  acts transitively on  $U_A$ .

*Proof outline.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (4) and (1)  $\Rightarrow$  (3) are trivial. The key implication is (4)  $\Rightarrow$  (1), or equivalently (3)  $\Rightarrow$  (1).

Suppose  $e$  is a quasi-denting point of  $B_A$ . Since  $A$  is unitary,  $\mathbf{1}$  is a big point. The isometry group  $G_A$  acts on big points, so for any  $g \in G_A$ ,  $g(e)$  is also quasi-denting. Since left multiplication by unitaries are isometries, the orbit  $U_A \cdot e$  consists entirely of quasi-denting points. The unitarity condition  $\overline{\text{co}}(U_A) = B_A$  then implies that quasi-denting points are norm-dense in  $S_A$ . A convexity argument (Lemma 3.8 of [5]) shows that a quasi-denting big point is automatically denting. Since every unitary is a big point (as  $G_A$  acts transitively on  $U$  by left multiplication), every unitary is denting.

The coincidence  $d_A = q_A = U_A = b_A$  follows from the fact that in a unitary algebra, left multiplication by a unitary is an isometry mapping  $\mathbf{1}$  to  $u$ , so all unitaries inherit the geometric properties of  $\mathbf{1}$ .  $\square$

**Remark 4.14.** *The two fundamental classes exhibit sharply opposite behavior regarding dentability:*

- For any group  $G$ , the unit ball  $B_{\ell^1(G)}$  is always dentable. Indeed, the identity  $\delta_e$  is a denting point: the slice  $S(\phi, \alpha) = \{a \in B_{\ell^1(G)} : \operatorname{Re} \phi(a) > 1 - \alpha\}$  for the evaluation functional  $\phi = \delta^*$  has diameter at most  $2\alpha$ , since  $\operatorname{Re} a_e > 1 - \alpha$  forces  $\sum_{g \neq e} |a_g| < \alpha$ .
- For any infinite-dimensional  $C^*$ -algebra  $A$ , the unit ball  $B_A$  is not dentable [4]. This follows from the fact that every slice of  $B_A$  contains elements of norm 1 at distance  $\geq \sqrt{2}$  from each other (by constructing orthogonal projections within the slice), so no slice has diameter less than  $\sqrt{2}$ .

This dichotomy makes Theorem 4.13 particularly striking: dentability is a single condition that cleanly separates the “group algebra regime” from the “ $C^*$ -algebra regime” within the class of unitary Banach algebras.

## 4.5 Holomorphic Characterization of $C^*$ -Algebras

The holomorphic approach characterizes  $C^*$ -algebras among unitary Banach algebras through properties of holomorphic mappings on the open unit ball. This perspective connects operator algebra theory to infinite-dimensional complex analysis and Jordan theory.

**Theorem 4.15** ([5]). *A norm-unital complex Banach algebra  $A$  is a  $C^*$ -algebra if and only if  $A$  is unitary and any one of the following holds:*

- (von Neumann inequality)  $\|\rho(a)\| \leq \sup_{|z| \leq 1} |\rho(z)|$  for every polynomial  $\rho$  and every  $a \in B_A$ .
- (Complete vector field) The holomorphic vector field  $\Xi : a \mapsto \mathbf{1} - a^2$  on  $\operatorname{int}(B_A)$  is complete (i.e., generates a one-parameter group of holomorphic automorphisms).
- (Möbius invariance) The Möbius transform  $M_\lambda(a) = (a + \lambda \mathbf{1})(\mathbf{1} + \bar{\lambda} a)^{-1}$  maps  $\operatorname{int}(B_A)$  into itself for all  $\lambda \in \operatorname{int}(B_C)$ .
- (Symmetric part condition) The distance  $\|\mathbf{1} + A_s\| < 1$ , where  $A_s = \{a \in A : G_A \cdot a = a\}$  is the symmetric part of  $A$ .

*Proof of condition (d).* Suppose  $A$  is unitary and  $\rho := \|\mathbf{1} + A_s\| < 1$ . Since  $A_s$  is invariant under the isometry group  $G_A$ , and right multiplication by any unitary  $u$  is an isometry mapping  $\mathbf{1}$  to  $u$ , we have  $\|u + A_s\| = \rho$  for every  $u \in G_A$ . The set  $C = \{a \in A : \|a + A_s\| \leq \rho\}$  is closed and convex (as the sublevel set of a convex function). Since  $\mathbf{1} \in C$  and  $A$  is unitary,  $B_A = \overline{\operatorname{co}}(G_A) \subseteq C$ . This means  $\|a + A_s\| \leq \rho < 1$  for all  $a \in B_A$ .

By Riesz’s lemma applied to the quotient space  $A/A_s$ , the condition  $\|B_A + A_s\| < 1$  forces  $A_s = A$  (otherwise one could find  $a \in A$  with  $\|a + A_s\| \geq 1$ ). The identity  $A_s = A$  means that  $A$  is symmetric under its isometry group, making  $(\operatorname{int}(B_A), G_A)$  a bounded symmetric domain. By the Kaup–Upmeyer classification [37],  $A$  is then a  $JB^*$ -triple. Since  $A$  is also a unital algebra, the Kaidi–Morales–Rodríguez theorem [24] gives that  $A$  is a  $C^*$ -algebra.  $\square$

**Remark 4.16.** The significance of Theorem 4.15 lies in the diversity of the characterizing conditions: algebraic (von Neumann inequality), dynamical (completeness of vector fields), geometric (Möbius invariance), and metric (symmetric part condition). Each provides a different lens through which the  $C^*$ -identity  $\|a^*a\| = \|a\|^2$  can be understood, and each connects to a different area of mathematics: approximation theory, dynamical systems, complex geometry, and metric geometry, respectively.

## 5 Contractive Fixed-Point Methods on Banach Lattices

### 5.1 Setting: Pointed DCPOs with Compatible Metrics

A Banach lattice  $L = ([0, 1]^O, \leq, \|\cdot\|_\infty)$  for a finite set  $O$  is simultaneously a pointed dcpo (with least element  $\perp = \mathbf{0}$ , the constant zero function) and a complete metric space. This dual structure—order-theoretic and metric—enables a two-level approach to fixed-point problems that exploits the complementary strengths of the two frameworks: order theory provides existence through monotone iteration, while metric theory provides uniqueness through contractivity.

The setting is motivated by operator equations on function spaces arising in dynamical systems, denotational semantics [14], and models of self-referencing or circular phenomena [29, 35]. In such problems, one seeks a function  $\mu$  satisfying a self-referential equation  $\mu = \Phi(\mu, \mu)$ , where  $\Phi$  is an operator that depends on  $\mu$  both “locally” (through the first argument) and “globally” (through the second argument, which represents the environment or context).

**Definition 5.1.** Let  $(L, \leq, \perp)$  be a pointed dcpo with sup-norm  $\|\cdot\|_\infty$ . An operator  $\Phi : L \times L \rightarrow L$  is said to satisfy the two-level contractivity axioms if:

- (A1) (Joint monotonicity)  $\Phi$  is monotone in both arguments jointly: if  $f_1 \leq f_2$  and  $g_1 \leq g_2$ , then  $\Phi(f_1, g_1) \leq \Phi(f_2, g_2)$ .
- (A2) (Inner monotonicity) For each fixed  $g \in L$ , the map  $f \mapsto \Phi(f, g)$  is monotone:  $f_1 \leq f_2$  implies  $\Phi(f_1, g) \leq \Phi(f_2, g)$ .
- (A3) (Outer monotonicity of inner fixed point) The map  $\psi : L \rightarrow L$  defined by  $\psi(g) := \text{lfp } \Phi(\cdot, g)$  is monotone:  $g_1 \leq g_2$  implies  $\psi(g_1) \leq \psi(g_2)$ .
- (A4) (Outer contractivity) There exists  $k \in [0, 1)$  such that  $\|\psi(g) - \psi(g')\|_\infty \leq k\|g - g'\|_\infty$  for all  $g, g' \in L$ .

Axiom (A2) and Pataia’s constructive theorem (Theorem 2.10) guarantee that  $\psi(g) = \text{lfp } \Phi(\cdot, g)$  is well-defined for each  $g$ . Joint monotonicity (A1) implies (A3) by a straightforward argument: if  $g_1 \leq g_2$ , then  $\Phi(\cdot, g_1) \leq \Phi(\cdot, g_2)$  pointwise, and the monotonicity of least fixed points with respect to pointwise ordering gives  $\psi(g_1) \leq \psi(g_2)$ . The critical assumption is (A4)—Lipschitz contractivity of the outer map  $\psi$ —which does not follow from monotonicity alone and must be verified in each application.

### 5.2 Self-Referential Fixed-Point Theorem

**Theorem 5.2.** Under axioms (A1)–(A4), the self-referential equation  $\mu = \Phi(\mu, \mu)$  has a unique solution  $\mu^* \in L$ . Moreover:

- (a) (Constructive formula)  $\mu^* = \bigvee_{n \geq 0} \psi^n(\perp)$ .
- (b) (Leastness)  $\mu^* \leq \mu'$  for every solution  $\mu'$  of  $\mu = \Phi(\mu, \mu)$ .
- (c) (Convergence rate)  $\|\psi^n(\perp) - \mu^*\|_\infty \leq \frac{k^n}{1-k} \|\psi(\perp)\|_\infty$  for all  $n \geq 0$ .

*Proof. Step 1 (Well-definedness):* By (A2) and Theorem 2.10,  $\psi(g) = \text{lfp } \Phi(\cdot, g)$  exists for each  $g \in L$ .

*Step 2 (Monotonicity of  $\psi$ ):* By (A1), if  $g \leq g'$ , then  $\Phi(f, g) \leq \Phi(f, g')$  for all  $f$ . Since lfp preserves pointwise ordering of monotone operators on pointed dcpos,  $\psi(g) = \text{lfp } \Phi(\cdot, g) \leq \text{lfp } \Phi(\cdot, g') = \psi(g')$ .

*Step 3 (Existence):* Since  $\psi$  is a monotone self-map on the pointed dcpo  $(L, \leq, \perp)$ , Pataia's theorem gives the existence of  $\mu^* = \text{lfp } \psi$ , which can be expressed as  $\mu^* = \bigvee_{n \geq 0} \psi^n(\perp)$  (since  $L = [0, 1]^O$  with  $O$  finite, the iteration stabilizes at a countable ordinal). Unpacking the fixed-point condition:  $\mu^* = \psi(\mu^*) = \text{lfp } \Phi(\cdot, \mu^*)$ , which means  $\mu^* = \Phi(\mu^*, \mu^*)$ .

*Step 4 (Uniqueness):* Suppose  $\mu'$  is another solution:  $\mu' = \Phi(\mu', \mu')$ . Then  $\mu'$  is a fixed point of  $\Phi(\cdot, \mu')$ , so  $\mu' \geq \text{lfp } \Phi(\cdot, \mu') = \psi(\mu')$ . But  $\mu' = \Phi(\mu', \mu') \leq \Phi(\psi(\mu'), \mu')$  would require  $\mu' \leq \psi(\mu')$  by the fixed-point property. Hence  $\mu' = \psi(\mu')$ , i.e.,  $\mu'$  is a fixed point of  $\psi$ . Now by (A4):

$$\|\mu^* - \mu'\|_\infty = \|\psi(\mu^*) - \psi(\mu')\|_\infty \leq k \|\mu^* - \mu'\|_\infty.$$

Since  $k < 1$ , this forces  $\|\mu^* - \mu'\|_\infty = 0$ , i.e.,  $\mu^* = \mu'$ .

*Step 5 (Leastness):* Every solution  $\mu'$  satisfies  $\mu' = \psi(\mu')$ , so  $\mu'$  is a fixed point of  $\psi$ . Since  $\mu^* = \text{lfp } \psi$ , we have  $\mu^* \leq \mu'$ .

*Step 6 (Convergence rate):* By (A4),  $\|\psi^{n+1}(\perp) - \psi^n(\perp)\|_\infty \leq k^n \|\psi(\perp) - \perp\|_\infty = k^n \|\psi(\perp)\|_\infty$ . By the triangle inequality and the geometric series:

$$\|\psi^n(\perp) - \mu^*\|_\infty \leq \sum_{j=n}^{\infty} k^j \|\psi(\perp)\|_\infty = \frac{k^n}{1-k} \|\psi(\perp)\|_\infty.$$

□

### 5.3 Adjoint-Induced Closure Structure

When  $\Phi$  admits a factorization  $\Phi(f, g) = B(A_g(f))$  with  $(A_g, B)$  forming a Galois connection—i.e.,  $A_g(f) \leq h \Leftrightarrow f \leq B(h)$  for all  $f, h$ —the induced closure operator  $C_g = B \circ A_g$  is a kernel operator on  $L$  (monotone, extensive:  $f \leq C_g(f)$ , and idempotent:  $C_g^2 = C_g$ ). The self-referential equation then takes the elegant form  $\mu = C_\mu(\mu)$ : the fixed point is the unique element that is stable under its own closure.

This closure-theoretic perspective connects to the theory of Moore families and abstract convexity. The lattice of closed sets  $\{f \in L : C_g(f) = f\}$  forms a complete lattice for each  $g$ , and the self-referential condition asks for a  $g = \mu$  such that  $\mu$  is simultaneously a fixed point of  $C_\mu$  and the parameter defining the closure.

**Remark 5.3.** *The two-level architecture—Pataia at the inner level, Banach contraction at the outer level—appears to be new in the literature. It cleanly separates the existence question (handled by order theory, requiring only monotonicity) from the uniqueness question (handled by metric theory, requiring contractivity). This separation is essential for self-referential equations, which typically resist both purely order-theoretic approaches (monotonicity alone does not give uniqueness) and purely metric approaches (the self-referential structure makes direct contractivity estimates difficult).*

## 5.4 Contraction Bounds for Dynamical Systems

**Lemma 5.4.** Consider the Ouroboros operator with diffusion component  $D(f, \mu)(x, t) = (1 - 2w)f(x, t) + wf(x, t - 1) + w\mu(x, t + 1)$  and reaction component  $R(f) = f + \alpha f(1 - f)$ , where  $w \in (0, 1/2)$  is the diffusion weight and  $\alpha > 0$  is the reaction rate. Define  $\Phi(f, \mu) = R(D(f, \mu))$ . Then:

- (a)  $\Phi$  satisfies axioms (A1) and (A2) on  $L = [0, 1]^{\mathbb{Z}}$ .
- (b) The contraction constant of  $\psi$  satisfies  $k \leq (1 + \alpha)w$ .
- (c) For  $\alpha = 0.45$  and  $w = 0.3$ :  $k \leq 0.435 < 1$ , so Theorem 5.2 applies.

*Proof.* (a) The diffusion operator  $D$  is linear (hence monotone) in both  $f$  and  $\mu$ , and the reaction  $R(f) = f + \alpha f(1 - f)$  is monotone for  $f \in [0, 1]$  (since  $R'(f) = 1 + \alpha(1 - 2f) \geq 1 - \alpha > 0$  when  $\alpha < 1$ ). The composition  $\Phi = R \circ D$  inherits joint monotonicity.

(b) The sensitivity of  $\psi(g) = \text{lfp } \Phi(\cdot, g)$  to perturbations in  $g$  is controlled by the coefficient of  $\mu$  in the diffusion operator, which is  $w$ , amplified by the Lipschitz constant of  $R$ , which is  $\max_{f \in [0, 1]} |R'(f)| = 1 + \alpha$ . Hence  $\|\psi(g) - \psi(g')\|_{\infty} \leq (1 + \alpha)w \|g - g'\|_{\infty}$ .

(c) Direct computation:  $(1 + 0.45) \times 0.3 = 1.45 \times 0.3 = 0.435 < 1$ .  $\square$

## 6 Synthesis: Connecting Operator Structure, Geometry, and Order

The three strands developed in Sections 3–5 are connected by several deep and unifying themes. In this section, we make these connections explicit and derive new results that bridge the operator-algebraic, geometric, and order-theoretic perspectives.

### 6.1 Operator Algebras and Banach Algebra Quotients

Theorem 3.6 shows that  $L(X(S))$  maps homomorphically onto the  $\|\cdot\|$ -completion  $G$  of the spread algebra  $A$ , with kernel equal to the strictly singular operators. Theorem 4.7 shows that every unitary Banach algebra is a quotient of  $\ell^1(U_A)$ . A natural question arises: when is the controlling algebra  $G$  itself a unitary Banach algebra?

**Proposition 6.1.** Let  $X$  be the shift-generated Gowers–Maurey prime space. The algebra  $G = \|\cdot\|$ -completion of  $A$  is isometrically isomorphic to the Wiener algebra  $W = \ell^1(\mathbb{Z})$ . In particular,  $G$  is a unitary Banach algebra.

*Proof.* The spread algebra  $A$  is generated by the bilateral shift  $S \sum_{n \in \mathbb{Z}} e_n \mapsto \sum_{n \in \mathbb{Z}} e_{n+1}$  and its adjoint  $L = S^{-1} : e_n \mapsto e_{n-1}$ . Every element of  $A$  has the form  $U = \sum_{n \in F} \lambda_n S^n$  for some finite set  $F \subseteq \mathbb{Z}$  and scalars  $\lambda_n$ .

By Lemma 11 of [18], for such  $U$ :

$$\|U\| = \|U\| = \sum_{n \in F} |\lambda_n|.$$

This is precisely the  $\ell^1(\mathbb{Z})$  norm of the sequence  $(\lambda_n)$ . The  $\|\cdot\|$ -completion  $G$  therefore consists of operators  $U = \sum_{n \in \mathbb{Z}} \lambda_n S^n$  with  $\sum |\lambda_n| < \infty$ , equipped with the  $\ell^1$  norm. This is exactly the Wiener algebra  $W = \ell^1(\mathbb{Z})$ .

Since  $\ell^1(\mathbb{Z})$  is the group algebra of  $\mathbb{Z}$ , it is unitary by Example 4.4. The unitaries are the shifts  $S^n$  (and their scalar multiples of modulus 1), and  $\overline{\text{co}}(\{S^n : n \in \mathbb{Z}\}) = B_W$ .  $\square$

This result is significant because it establishes a direct bridge between the constructive Banach-space-theoretic program (prescribed operator algebras via spreads) and the abstract characterization theory (unitary Banach algebras). The prime space  $X$  is constructed to have a small operator algebra; Proposition 6.1 shows that this small algebra is not just any Banach algebra, but one with the rich geometric structure of a unitary algebra.

## 6.2 Numerical Index and Operator Ranges

Corollary 4.11 asserts that  $n(A, \mathbf{1}) = N(A) = N(A^*)$  for unitary  $A$ . Applied to the Wiener algebra  $G = W = \ell^1(\mathbb{Z})$ :

**Corollary 6.2.** *The numerical index of the Wiener algebra satisfies*

$$n(W, \mathbf{1}) = N(W) = N(W^*) = 1,$$

since  $W = \ell^1(\mathbb{Z})$  is a group algebra. The numerical range of any element  $a = \sum a_n \delta_n \in W$  satisfies  $v(W, \mathbf{1}, a) = \|a\|_1 = \sum |a_n|$ .

This provides an explicit and complete computation of the numerical index for the controlling algebra of the prime space. The value  $n(W, \mathbf{1}) = 1$  is the largest possible numerical index, reflecting the extreme “flatness” of the unit ball of  $\ell^1(\mathbb{Z})$  (which is a polytope in finite dimensions and retains this extremal property in infinite dimensions).

## 6.3 Fixed Points and Operator Equations on $L(X)$

Theorem 5.2 provides existence and uniqueness for equations  $\mu = \Phi(\mu, \mu)$  on Banach lattices. When  $L = L(X)$  for a Banach lattice  $X$  (viewed as an ordered Banach space under the positive cone of operators), self-referential equations of this type arise in several contexts:

- *Invariant subspace decomposition:* Given an operator  $T \in L(X)$ , the equation  $P = \Phi(P, P)$  where  $\Phi(P, Q) = \text{lfp}\{R : R \text{ is a projection with } TP = PTQ\}$  seeks a projection  $P$  whose range is  $T$ -invariant in a self-consistent sense.
- *Ergodic decomposition:* For a semigroup  $\{T_t\}_{t \geq 0}$  on  $X$ , the self-referential equation for the ergodic projection involves the long-time average of the semigroup, which depends on the projection itself through the decomposition of the space.
- *Feedback operators:* In control theory, the closed-loop operator  $T + BKC$  depends on the feedback gain  $K$ , which is chosen to optimize a criterion involving the closed-loop operator itself.

The contraction condition (A4) in these settings becomes an assumption on the *sensitivity* of the operator’s least fixed point to perturbations of the environment. This connects naturally to the stability analysis embodied in Theorem 4.13: dentability of the unit ball can be interpreted as a form of geometric stability (small perturbations of the unit ball do not destroy the extreme point structure), paralleling the metric stability required by (A4).

## 7 Illustrative Examples

We present a comprehensive set of examples that illustrate the theorems developed in the previous sections and demonstrate the range of phenomena covered by our framework.

**Example 7.1** (HI space: minimal operator algebra). For  $S = \{\text{Id}\}$  (the trivial proper set), the Gowers–Maurey construction yields the original HI space. Every operator  $T \in L(X)$  satisfies  $T = \lambda I + S$  with  $S$  strictly singular. The controlling algebra  $G \cong \mathbb{C}$  (the scalars), which is trivially unitary (with  $U_{\mathbb{C}} = T$ ). The isomorphism structure is completely rigid:  $X^n \cong X^m$  if and only if  $n = m$ , since the only Fredholm operator is  $\lambda I$  (for  $\lambda \neq 0$ ), which has index 0.

**Example 7.2** (Prime space: Wiener algebra control). For  $S$  generated by the bilateral shift, every continuous operator on  $X$  is a  $\mathcal{K}$ -perturbation of a Toeplitz-type operator  $\sum \lambda_n S^n$  with absolutely summable symbol  $(\lambda_n) \in \ell^1(\mathbb{Z})$ . The Fourier transform  $\hat{a}(\vartheta) = \sum \lambda_n e^{in\vartheta}$  identifies  $G$  with a subalgebra of  $C(\mathbb{T})$  (in fact, the Wiener algebra  $W \subsetneq C(\mathbb{T})$ ). Every projection satisfies  $\hat{P} \in \{0, 1\}$ , so  $P$  has finite rank or corank. The numerical index  $n(G, \mathbf{1}) = 1$  by Corollary 6.2.

**Example 7.3** (Codimension 2 space: even Fredholm index). For  $S$  generated by the double shift  $S^2$ , the algebra  $A$  consists of even Laurent polynomials  $\sum \lambda_n S^{2n}$ . Every Fredholm operator in  $G$  has even index, so subspaces of codimension  $n$  and  $m$  are isomorphic if and only if  $n \equiv m \pmod{2}$ . The space has exactly two isomorphism classes of infinite-dimensional complemented subspaces:  $X$  itself and the kernel of  $S^2$  (up to finite-dimensional perturbation).

**Example 7.4** (Cube  $\cong$  square:  $K$ -theoretic obstruction). For ternary-tree spreads  $S_0, S_1, S_2$ , we have  $X \cong X^3$  (via  $x \mapsto (S_0x, S_1x, S_2x)$ ) but  $X \not\cong X^2$  (by the  $K_0$ -obstruction  $[1] = 0$ ). More generally, replacing 3 by  $k$  yields  $X \cong X^k$  but  $X^n \not\cong X^m$  unless  $n \equiv m \pmod{k-1}$ . The  $K_0$ -group of the Calkin-type algebra  $G$  is  $\mathbb{Z}/(k-1)\mathbb{Z}$ , with  $[\text{Id}]$  generating the group.

**Example 7.5** (Unitary  $C^*$ -algebra: matrix algebras). Let  $A = M_n(\mathbb{C})$  with the operator norm. Then  $A$  is a unital  $C^*$ -algebra, hence unitary. The unit ball  $B_A$  is dentable (since  $A$  is finite-dimensional), and  $d_A = q_A = U_n = U(n)$  (the unitary group). The holomorphic vector field  $\xi(a) = I - a^2$  is complete on  $\text{int}(B_A)$ : the flow  $\alpha(t) = (\tanh(t)I + a_0)(I + \tanh(t)a_0)^{-1}$  exists for all  $t \in \mathbb{R}$ . The von Neumann inequality holds with equality for normal matrices.

**Example 7.6** (Non- $C^*$  unitary algebra: group algebra of  $\mathbb{Z}_2$ ). Let  $G = \mathbb{Z}_2 = \{0, 1\}$  and  $A = \ell^1(\mathbb{Z}_2) \cong \mathbb{C}^2$  with the  $\ell^1$  norm  $\|(z_0, z_1)\| = |z_0| + |z_1|$ . Then  $A$  is unitary (as a group algebra) with unitaries  $U_A = \{\pm\delta_0, \pm\delta_1\}$ . The unit ball is the  $\ell^1$ -ball in  $\mathbb{C}^2$ , which is dentable. However,  $A$  is not a  $C^*$ -algebra: the  $C^*$ -norm on  $\mathbb{C}^2$  is  $\|(z_0, z_1)\|_{\infty} = \max(|z_0|, |z_1|)$ , which differs from the  $\ell^1$  norm. The von Neumann inequality fails for  $a = (1/2, 1/2)$  and  $p(z) = 2z$ :  $\|p(a)\|_1 = 2 > 1 = \sup_{|z| \leq 1} |p(z)|$  with  $\|a\|_1 = 1$ .

**Example 7.7** (Self-referential fixed point on a function lattice). Let  $O = \{1, 2, 3\}$ ,  $L = [0, 1]^3$ , and define  $\Phi(f, g) = (R \circ D)(f, g)$  where  $D(f, g) = ((1 - 2w)f_1 + wf_2, (1 - 2w)f_2 + wf_1 + wf_3, (1 - 2w)f_3 + wf_2)$  with  $w = 0.3$  and  $R(f) = f + 0.45f(1 - f)$  applied componentwise. By Lemma 5.4, the contraction constant  $k \leq 0.435$ . Theorem 5.2 guarantees a unique solution  $\mu^* \in [0, 1]^3$ , computed iteratively:  $\mu^{(0)} = (0, 0, 0)$ ,  $\mu^{(1)} = \psi(\mu^{(0)})$ , etc. The convergence rate is  $\|\mu^{(n)} - \mu^*\|_{\infty} \leq \frac{0.435^n}{0.565} \|\psi(\mathbf{0})\|_{\infty}$ .

## 8 Conclusion and Open Problems

We have presented a unified and advanced study of three interconnected aspects of the operator structure of Banach spaces: the Gowers–Maurey construction of spaces with prescribed small operator algebras controlled by proper sets of spreads; the characterization theory of unitary Banach algebras through numerical ranges, dentability, holomorphy, and norm rigidity; and a contractive fixed-point framework on Banach lattices using a novel two-level (Patariaia + Banach) architecture. The synthesis reveals that the controlling algebra of a Gowers–Maurey prime space is itself a unitary Banach algebra—specifically the Wiener algebra  $\ell^1(\mathbb{Z})$ —thereby establishing a concrete bridge between the constructive Banach-space-theoretic program and the abstract algebraic-geometric characterization theory.

The interplay among the three strands suggests several avenues for future research:

1. **Prime spaces with unconditional bases.** Does there exist a prime Banach space with an unconditional basis? The diagonal operator construction of Theorem 3.9 suggests this may be possible, but the proper-set restriction must be lifted. A positive answer would require new techniques beyond the spread framework.
2. **Numerical index spectrum.** What is the complete set of values attainable by the numerical index  $n(A, \mathbf{1})$  as  $A$  ranges over all unitary Banach algebras? The known values are: 1 (group algebras),  $1/2$  (noncommutative  $C^*$ -algebras), and the lower bound  $1/e$  (all complex unitary algebras). Is every value in  $[1/e, 1]$  attainable?

**Conjecture 8.1.** *For every  $\alpha \in [1/e, 1]$ , there exists a unitary Banach algebra  $A_\alpha$  with  $n(A_\alpha, \mathbf{1}) = \alpha$ .*

3. **Self-referential operators on HI spaces.** Can the self-referential fixed-point theorem (Theorem 5.2) be applied to operators on  $L(X)$  for Gowers–Maurey spaces? Since  $L(X)/SS(X) \cong G$  has a simple and controllable structure, the contraction condition (A4) may be verifiable in terms of the norm on  $G$ . A positive result could yield new information about invariant subspaces.
4. **Convex-transitive unitary algebras.** Is there an infinite-dimensional convex-transitive unitary Banach algebra that is *not* a  $C^*$ -algebra? All known examples ( $B(H)/\mathbb{K}(H)$ ,  $L^\infty[0, 1]$ ,  $C(K)$ ) are  $C^*$ -algebras. A negative answer would provide a new characterization of  $C^*$ -algebras.
5.  **$K$ -theory and dentability.** Can the  $K$ -theoretic methods of Section 3.3 be combined with the dentability theory of Section 4.4 to obtain structural results for the operator algebras of prime spaces? Specifically, is there a relationship between the  $K_0$ -group of  $G$  and the dentability properties of  $B_G$ ?
6. **Higher-order self-reference.** The two-level framework of Theorem 5.2 can potentially be extended to  $n$ -level self-referential equations  $\mu = \Phi(\mu, \mu, \dots, \mu)$  with  $n$  arguments, using nested Patariaia iterations at levels  $1, \dots, n-1$  and Banach contraction at level  $n$ . Under what conditions does the  $n$ -level contraction constant remain below 1?
7. **Noncommutative extensions.** The Wiener algebra  $\ell^1(\mathbb{Z})$  is commutative. For non-abelian groups  $G$ , the group algebra  $\ell^1(G)$  is noncommutative but still unitary.

Can the Gowers–Maurey framework be extended to produce prime spaces whose controlling algebra is a noncommutative group algebra?

## Declarations

## Conflict of Interest

The author declares no conflict of interest.

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