ON A FUNCTION CLASS RELATED TO COMPLETELY MONOTONE FUNCTIONS

S. GUO

College of Global Talents, Beijing Institute of Technology, Zhuhai
Zhuhai City, Guangdong Province, 519088, P. R. China.
Email: 19013@bitzh.edu.cn

ABSTRACT

In this review article, we introduce some basic knowledge closely related to completely monotone functions.

2010 Mathematics Subject Classification: 33E20, 33E30

Keywords: absolutely monotone functions, completely monotone sequences

---

1. Absolutely Monotone Functions

Let's first introduce the notion of absolutely monotone functions, which is closely associated with that of completely monotone functions.

Bernstein [1] in 1914 first introduced

Definition A. A function f is said to be absolutely monotone on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Here, $C(I)$ is the set of all continuous functions on the interval I, and I^o is the interior of the interval I.

We use $AM(I)$ to denote the class of all absolutely monotone functions on the interval I.

For the interval [a, b) or [a, b], Bernstein [1] also gave an equivalent definition to Definition A as follows:

Definition B. A function f is absolutely monotone on the interval [a, b) if and only if

$$\Delta_h^n f(x) := \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f(x+kh) \geq 0$$

for all $n \in \mathbb{N}_0$ and for all x and h such that

$$a \leq x \leq x + nh < b.$$

If, in addition,

$$f(b) = f(b-),$$

then f is called absolutely monotone on the interval $[a, b]$.

Here, and throughout the paper, $\mathbb{N}_0$ is the set of all non-negative integers.

In 1935 Gruss introduced the following

Definition C. A function f is absolutely monotone on the interval [0, 1] if f is continuous there and if for all $n \in \mathbb{N}$

$$\Delta^k f\left(\frac{i}{n}\right) \geq 0, \quad k = 0, 1, \cdots, n; \quad i = 0, 1, \cdots, n-k. \tag{1}$$

Here

$$\Delta^k f(x) := \Delta^{k-1} f\left(x + \frac{1}{n}\right) - \Delta^{k-1} f(x),$$

and

$$\Delta^0 f(x) := f(x).$$

Here, and throughout the paper, $\mathbb{N}$ is the set of all positive integers.

It is easy to modify this definition to apply to the interval $[a, b]$.

Since

$$\Delta^{k-1} f(x) = \Delta^k f\left(x - \frac{1}{n}\right) + \Delta^{k-1} f\left(x - \frac{1}{n}\right),$$

we may replace the condition (1) with the following one

$$\Delta^k f(0) \geq 0, \quad k = 0, 1, \cdots, n. \tag{2}$$

For the interval [0, 1] (then [a, b]), Gruss' definition C is equivalent to Bernstein's definition.

That Bernstein's definition implies that of Gruss is obvious. For the converse part, first we need the following well-known

Theorem D. Suppose that $f \in C[0,1]$, then $B_n f$ converges to f uniformly on $[0,1]$. Here $B_n f$ is the Bernstein Polynomial of f, i.e.

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Secondly, from this result we can show

Theorem E. $f \in C[0,1]$ satisfies the condition (2) if and only if f is the uniform limit of a sequence of polynomials with nonnegative coefficients.

Then by use of Theorem E, we can prove that Gruss' definition C does imply Bernstein's definition B.

For details of this part and the proof of the equivalence of Bernstein's two definitions, see [9, Chapter IV].

Clearly, if

$$f, g \in AM(I),$$

then

$$\alpha f + \beta g \in AM(I) \text{ for } \alpha, \beta \geq 0,$$

And

$$fg \in AM(I)$$

by using of Leibniz's rule.

From the definition, a convergent series of powers of $(x - a)$ with nonnegative coefficients represents an absolutely monotone function on $[a, a + \rho)$, where ρ is the radius of convergence of the power series.

On the other hand, suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, |x-a| < \rho.$$

If any coefficient of the series is negative, then $f(x)$ can not be absolutely monotone on any left or right neighborhood of $x = a$. If any coefficient of the series is zero, then $f(x)$ can not be absolutely monotone on any left neighborhood of $x = a$ unless $f(x)$ is a polynomial.

Indeed, if $a_k < 0$, then

$$f^{(k)}(a) < 0.$$

Hence

$$f^{(k)}(x) < 0$$

in some left and in some right neighborhood of a.

If $a_k = 0$, then

$$f^{(k)}(a) = 0.$$

If $f(x)$ is absolutely monotone on a left neighborhood of a, then $f^{(k)}(x) \equiv 0$ there since $f^{(k)}(x)$ is increasing. And this means that $f(x)$ is a polynomial.

We also notice

Theorem F. Any function which can be expressed as a series of powers of (x−a) must be the difference of two functions which are absolutely monotone on a right neighborhood of a.

This result follows from the observation

$$\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} |a_n| \, (x-a)^n - \sum_{n=0}^{\infty} (|a_n| - a_n)(x-a)^n.$$

One of the important properties of absolutely monotone functions is that it is analytic or holomorphic. More precisely we have

Theorem G. If $f(x)$ is absolutely monotone on $[a, b)$, then it can be extended analytically into the region of the complex z - plane: $|z-a| < b-a$, where $z = x + iy$.

From this result, an absolutely monotone function on $[a, \infty)$ can be extended as an entire function.

Now suppose that $f(x)$ is absolutely monotone on (a, b) and

$$f(c) = 0$$

for some $c \in (a, b)$.

Since $f(x)$ is nonnegative and increasing on (a, b), we see that

$$f(x) = 0$$

for all $x \in (a, c)$. This fact along with Theorem G leads to

Theorem H. Suppose that $f \in AM(I)$. If there exists $x_0 \in I^o$ such that

$$f(x_0) = 0,$$

then $f(x) \equiv 0$ on I.

2. Completely Monotone Sequences

Another concept which is related to completely monotone functions is the notion of a completely monotone sequence.

Definition I ([9, Chapter III]**).** A sequence $\{\mu_n\}_0^\infty$ is called completely monotone if

$$(-1)^k \Delta^k \mu_n \geq 0, \quad n, k \in \mathbb{N}_0,$$

where

$$\Delta^0 \mu_n = \mu_n, \quad \Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n.$$

Such a sequence is called totally monotone in [11].

By the principle of mathematical induction, we can prove that for $n, k \in \mathbb{N}_0$,

$$\Delta^k \mu_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \mu_{n+k-i} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i} \mu_{n+i}.$$

In 1963, Lorch and Moser [6] showed that for a completely monotone sequence $\{\mu_n\}_0^\infty$, we always have

$$(-1)^k \Delta^k \mu_n > 0, \quad n, k \in \mathbb{N}_0$$

unless

$$\mu_n = c,$$

a constant for all $n \in \mathbb{N}$.

Hausdorff [5] in 1921 proved a fundamental result for such sequences

Theorem J. A sequence $\{\mu_n\}_0^\infty$ is completely monotone if and only if there exists an increasing function $\alpha(t)$ on $[0,1]$ such that

$$\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0. \tag{3}$$

In 1931 Widder [10] gave

Definition K. A sequence $\{\mu_n\}_0^\infty$ is called minimal completely monotone if it is completely monotone and if it will not be completely monotone when μ_0 is replaced by a number less than μ_0.

For such a class of sequences, Widder proved [10]

Theorem L. A sequence $\{\mu_n\}_0^\infty$ is minimal completely monotone if and only if there exists an increasing function $\alpha(t)$ on $[0,1]$ with

$$\alpha(0) = \alpha(0+)$$

such that

$$\mu_n = \int_0^1 t^n d\alpha(t), \quad n \in \mathbb{N}_0.$$

Apparently not all completely monotone sequences are minimal.

Guo showed the following two results in [4] among others

Theorem M. For each completely monotone sequence $\{\mu_n\}_0^\infty$, there exists one and only one number μ_0^* such that

$$\{\mu_0^*, \mu_1, \mu_2, \cdots\}$$

is minimal completely monotone.

Theorem N. Suppose that the sequence $\{\mu_n\}_0^\infty$ is completely monotonic, then for any $m \in \mathbb{N}$ the sequence $\{\mu_n\}_m^\infty$ is minimal completely monotonic.

Absolutely monotone functions and completely monotone sequences are closely related to completely monotone functions which are important function class in the theory and applications of special functions.

REFERENCES

1. S. Bernstein, Sur la definition et les proprietes des fonctions analytiques dune variable reelle, Math. Ann. 75 (1914), 449–468.
2. W. Feller, Completely monotone functions and sequences, Duke Math. J. 5 (1939), 661–674.
3. G. Gruss, Bemerkungen zur theorie der voll-bzw. Mehrfachmonotonen funktionen, Math. Z. 39 (1935), 732–741.
4. S. Guo, Some properties of completely monotonic sequences and related interpolation, Appl. Math. Comp. 219(10) (2013), 4958-4962.
5. F. Hausdorff, Summationsmethoden und momentfolgen I, Math. Z. 9 (1921), 74–109.
6. L. Lorch and L. Moser, A remark on completely monotonic sequences, with an application to summability, Canad. Math. Bull. 6 (1963),171–173.
7. L. Lorch and D. J. Newman, On the composition of completely monotonic functions, completely monotonic sequences and related questions, J. London Math. Soc.(2) 28 (1983), 31–45.
8. C. O'Cinneide, A property of completely monotonic functions, J. Austral. Math. Soc. Ser. A 42 (1987), 143–146.
9. D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, New Jersey, 1941.
10. D. V. Widder, Necessary and sufficient conditions for the representation of a function as a Laplace integral, Trans. Amer. Math. Soc. 33 (1931), 851–892.
11. J. Wimp, Sequence transformations and their applications, Academic Press, New York, 1981.