International Journal of Engineering and Techniques - Volume 12 Issue 3, May-June - 2026

Fuzzy Extensions of Tri-Ideals in Semigroup

RAVI SRIVASTAVA[1], NEHA AHUJA[2] and VIVEK SHARMA[3]

[1] *Department of Mathematics, Swami Shraddhanand College,*
University of Delhi, Alipur, Delhi-110036 (INDIA).
E-mail: rksri@ss.du.ac.in

[2] *Department of Mathematics, Swami Shraddhanand College,*
University of Delhi, Alipur, Delhi-110036 (INDIA).
E-mail: neha86ahuja@gmail.com

[3] *Research Scholar, Swami Shraddhanand College,*
University of Delhi, Delhi-110007 (INDIA).
E-mail:vivek8024sharma@gmail.com

Abstract: Fuzzy mathematics, grounded in the framework of Fuzzy Logic, provides a robust methodology for modelling and analyzing systems characterized by uncertainty, vagueness, and partial truth values, which are not adequately addressed by classical binary logic. Its practical significance is evident across diverse real-world applications, including adaptive control in household appliances, intelligent traffic management, advanced vehicular systems, and performance optimization in consumer electronics. Furthermore, fuzzy-based approaches contribute to decision-support mechanisms in healthcare diagnostics, financial risk evaluation, and meteorological analysis. These applications highlight its alignment with human reasoning processes, where gradations of truth are more natural than crisp dichotomies.

In this context, the present study formulates pointwise definitions of fuzzy left, right, and tri-ideals in semigroups and rigorously demonstrates their equivalence with established set-theoretic representations. The investigation is extended to semigroups through the application of Tom Head's metatheorem, which facilitates the systematic derivation of fuzzy counterparts of classical algebraic results in a simplified and computation-free manner. The approach yields concise and transparent proofs, thereby enhancing theoretical clarity. Additionally, the structural properties of fuzzy tri-ideals are examined in both simple and regular semigroups, with particular emphasis on their closure under projection. The study establishes several key results, including the preservation of tri-ideal structure under intersections and products involving various classes of fuzzy ideals and subsemigroups. It is also shown that all conventional ideal types—left, right, two-sided, bi-, interior, and quasi-ideals—naturally satisfy the conditions of tri-ideals within a semigroup. Finally, necessary and sufficient conditions are derived to characterize when a fuzzy subsemigroup qualifies as a fuzzy tri-ideal, along with criteria for the regularity of semigroups in the framework of fuzzy tri-ideal theory.

Keywords:

Semigroups; fuzzy tri-ideals; Rep function;
metatheorem; projection closed; regular semigroup.

2020 Mathematics Subject Classification Number-
20M10.

1. Introduction

The notion of bi-ideals in semigroups and associated ring structures was first formulated by Good and Hughes in [1] 1952, and was subsequently further developed by Lajos and Szasz [7]. The concept of interior ideals in semigroups was later introduced by Szasz [12]. Subsequently, Kuroki extended these ideas to the fuzzy setting through the introduction of fuzzy bi ideals and fuzzy interior ideals in semigroups. The study of generalized ideals within algebraic systems has emerged as an important direction in abstract algebra, contributing significantly to a deeper structural understanding of these systems. One-sided ideals constitute an extension of classical ideal theory, while quasi-ideals provide a broader formulation that incorporates both left and right ideal characteristics. Bi- ideals represent a further abstraction of quasi-ideals by relaxing additional structural constraints. Expanding this hierarchy, Rao [8, 9] introduced the notion of tri-ideals as a unifying generalization of left, right, two sided, bi, quasi, and interior ideals in semigroups and semirings within the classical algebraic setting.

In 1995, Tom Head [2] proposed a metatheorem that enables the extension of results formulated for crisp algebraic substructures to analogous statements in the context of fuzzy substructures. This contribution represented an important progression in the theoretical development of fuzzy algebra, as it established a systematic framework for transferring classical structural properties into fuzzy settings. As a result, conclusions concerning substructures in semigroups, rings, semiring, and lattices can be derived concisely from the analogous results for crisp substructures. Indeed, the study presented in [1] establishes a clear and systematic conceptual approach for extending results from classical crisp structures to their fuzzy counterparts whenever such a transfer is feasible. Subsequent works, [4,10] adopted this metatheorem in the context of fuzzy algebra. More recently, Srivastava and Arvind [11] employed this theorem to generalize results concerning ideal theoretic structures in semirings, thereby extending their applicability to the framework of fuzzy algebras. The application of the metatheorem further yielded alternative demonstrations for a variety of established results, thereby reducing reliance on extensive computational derivations.

In this paper, we begin by presenting the point-wise definition of fuzzy (left tri-, right tri-, tri-) ideals in semigroups. The equivalence of these definitions with the set-theoretic definitions provided by Rao [8] is demonstrated. By employing the metatheorem, we extend the results concerning this notion in semigroups to the fuzzy context. Additionally, we investigate this notion within semigroups and analyze their properties using the metatheorem. The characterization of regular semigroups through these concepts, utilizing the metatheorem, is also explored.

2. Preliminaries

A semigroup is defined as an algebraic structure $(S, *)$, where S is a nonempty set equipped with an associative binary operation $*$. An element $a \in S$ is said to be regular whenever there exists an element $b \in S$ satisfying the relation $a = aba$. A semigroup is regarded as regular if each of its elements fulfills this condition. Furthermore, a semigroup S is described as left simple or right simple when it admits no nontrivial left ideals or right ideals, respectively. If S contains no proper two-sided ideals, then it is characterized as a simple semigroup. For basic definitions and results in semigroup we refer [8-9].

Definition 2.1. [9] A non-empty set A of a semigroup S is called a left (right) tri-ideal of S if $ASAA \subseteq A$ ($AASA \subseteq A$) and is called tri-ideal of S if it is both a left and a right tri-ideal of S.

Zadeh [13], defined a fuzzy subset f of a non-empty set X as a function $\gamma : X \rightarrow [0, 1]$. Let S be a semigroup. Throughout the paper *F(S)* stands for set of all fuzzy subset of *S*. Throughout this paper *J* will stands for [0,1). Also $C_{ss}, (C_l, C_r, C_i, C_{in}, C_b, C_q)$ denotes the classes of crisp subsemigroups, left (right, two sided, bi-, interior ideals and quasi-) ideals of a semigroup *S* and $\boldsymbol{C}_{ss}, (\boldsymbol{C}_l, \boldsymbol{C}_r, \boldsymbol{C}_i, \boldsymbol{C}_{in}, \boldsymbol{C}_b, \boldsymbol{C}_q)$ denotes their respective fuzzy classes.

Definition 2.2. [5, 6] A fuzzy set γ of a semigroup S is called a

(i) fuzzy subsemigroup of S if $\gamma(xyz) \geq \min\{\gamma(x), \gamma(y), \gamma(z)\} \ \forall\, x, y, z \in S.$

(ii) fuzzy left (right) ideal of S if $\gamma(xy) \geq \gamma(y) \ (\gamma(xy) \geq \gamma(x)) \ \forall\ x, y \in S.$

(iii) fuzzy ideal of S if $\gamma(xy) \geq \gamma(y)$ and $\gamma(xy) \leq \gamma(x) \ \forall\ x, y \in S.$

(iv) fuzzy bi-ideal of S if $\gamma \in C_{ss}$ and $\gamma(xyz) \geq \min\{\gamma(x), \ \gamma(z)\} \ \forall\, x, y, z \in S.$

(v) fuzzy interior ideal of S if $\gamma \in C_{ss}$ and $\gamma(xyz) \geq \gamma(y) \ \forall\, x, y, z \in S.$

Definition 2.3. [10] A fuzzy set γ of a semigroup S is called a fuzzy quasi-ideal of S if $\min\{\sup_{z=xy}(\gamma(x), \ \sup_{z=xy}\gamma(y)\} \leq \gamma(z) \ \forall z \in \mathrm{S}$

Definition 2.4. A fuzzy set γ of a semigroup S is called a

(i) fuzzy left tri- ideal of S if $\gamma \circ S \circ \gamma \circ \gamma \subseteq \gamma$

(ii) fuzzy right tri-ideal of S if $\gamma \circ \gamma \circ S \circ \gamma \subseteq \gamma$.

(iii) fuzzy tri-ideal of S if $\gamma \circ S \circ \gamma \circ \gamma \subseteq \gamma$ and $\gamma \circ \gamma \circ S \circ \gamma \subseteq \gamma$.

Now we briefly study 'metatheorem' formulated by Tom Head [3] in the year 1995. Let S be a ternary semigroup and P(S) and C(S) denotes the set of all subsets and characteristic subsets of S. The mapping $Chi: P(S) \to C(S)$ defined by $Chi(A) = \chi_A$ is a bijection. It can be seen easily that $P(S) \cong C(S)$ under the isomorphism *Chi* and *Chi* commutes with the finite intersection and product of sets in a semigroup S.

Definition 2.5 [2] If $\gamma \in F(S)$ and $r \in J = [0, 1)$. Then the function Rep: $F(S) \to C(S)^J$ is defined by

$$\mathrm{Rep}(\gamma)(r)(x) = \begin{cases} 1 & \text{if } \gamma(x) > r \\ 0 & \text{if } \gamma(x) \leq r \end{cases}$$

Proposition 2.1 [2] (i) The function Rep is injective.

(ii) $\mathrm{Rep}\left(\bigcap_{i=1}^{k} \mu_i\right) = \bigcap_{i=1}^{k} \mathrm{Rep}(\mu_i)$ and

(iii) $\mathrm{Rep}\left(\bigcup_{i \in M} \mu_i\right) = \bigcup_{i \in M} \mathrm{Rep}(\mu_i)$, where $\mu_i \in F(S)$ and M is an index set.

Proposition 2.2. [2] Rep is an order isomorphism of $F(S)$ onto $I(S)$, where $I(S)$ denotes the image of the Rep function.

Definition 2.10. Let $\gamma_1, \ \gamma_2 \ \in F(S)$.

$$(\gamma_1 * \gamma_2)(x) = \begin{cases} \sup_{x = x_1 * x_2} \ [\min\{\gamma_1(x_1), \ \gamma_2(x_2)\}] & \\ 0 & \text{if } x \text{ not express} \end{cases}$$

Proposition 2.3. [2] For $\gamma_1, \gamma_2 \in F(S)$, $\mathrm{Rep}(\gamma_1 * \gamma_2) = \mathrm{Rep}(\gamma_1) * \mathrm{Rep}(\gamma_2)$.

Definition 2.6.[2] Consider $\mathcal{D}$ as a collection of fuzzy sets within a semigroup S. We defined $\mathcal{D}$ as being projection closed if every $\gamma \in \mathcal{D}$ and for any $r \in J$, the Rep(γ)(r) remains an element of $\mathcal{D}$.

Proposition 2.4. [2] $D_1, D_2 (\mathcal{D}_1, \mathcal{D}_2)$ be the classes of crisp (fuzzy) subsets of a semigroup S. Then $\mathcal{D}_1 \subseteq \mathcal{D}_2 (\mathcal{D}_1 = \mathcal{D}_2) \Leftrightarrow D_1 \subseteq D_2 (D_1 = D_2)$.

Metatheorem 2.5. [2] Let $(S, *)$ be a semigroup. Consider the algebra- $(F(S), \text{inf}, \text{sup}, *)$. Let $L(a_1, a_2, ..., a_m)$ and $M(a_1, a_2, ..., a_m)$ be two expressions defined over the variables set $\{a_1, a_2, ..., a_m\}$ and operations set $\{ \text{inf}, \text{sup}, +\}$ on $P(S)$. Let $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_m$ are projection closed classes of fuzzy sets of S and $D_1, D_2, ..., D_m$ be their crisps classes respectively.

$$L(\gamma_1, \gamma_2, ..., \gamma_m) \ \text{REL} \ M(\gamma_1, \gamma_2, ..., \gamma_m)$$

holds $\forall \gamma_1 \in \mathcal{D}_1, \dots, \gamma_m \in \mathcal{D}_m \Leftrightarrow$ it holds $\forall \gamma_1 \in D_1, \dots, \gamma_m \in D_m$ where REL represent one of the three symbols $\leq$, = or $\geq$.

3. Fuzzy Tri-Ideals in Semigroups

In this section, we begin by presenting the point-wise definitions of fuzzy left (right) tri-ideals within a semigroup and demonstrate that these definitions are equivalent to those provided in [6].

Definition 3.1. A fuzzy set γ of a semigroup S is called a fuzzy left tri-ideal if $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(z), \gamma(w)\}\ \forall x, y, z, w \in S$.

Definition 3.2. A fuzzy set γ of a semigroup S is called a fuzzy right tri-ideal if $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(y), \gamma(w)\}\ \forall x, y, z, w \in S$.

Definition 3.3. A fuzzy set γ of a semigroup S is called a fuzzy tri-ideal if $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(z), \gamma(w)\}$ and $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(y), \gamma(w)\}\quad \forall x, y, z, w \in S$

Now we provide the equivalence between the Definitions 2.4 and 3.2 of fuzzy left tri-ideals in a semigroup.

Theorem 3.4. Le $\gamma \in F(S)$. Then, $\gamma \in \mathcal{C}_{til} \Leftrightarrow \gamma \circ S \circ \gamma \circ \gamma \subseteq \gamma$.

Proof: Let $z \in S$.

Then, $\gamma \circ S \circ \gamma \circ \gamma \subseteq \gamma$

$\Leftrightarrow \quad (\gamma \circ S \circ \gamma \circ \gamma)(z) \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min\{\gamma \circ S(uv),\ \gamma \circ \gamma(xy)\}] \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min(\gamma(u), S(v), \gamma(x), \gamma(y))] \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min(\gamma(u),\ \gamma(x), \gamma(y))] \leq \gamma(z)$

$\Leftrightarrow \quad \gamma(uvxy) \geq \{\min[\gamma(u),\ \gamma(x),\ \gamma(y)]\}\ \forall\ u, v, x, y \in$

$\Leftrightarrow \quad \gamma \in \mathcal{C}_{til}$.

Equivalence between the Definitions 2.4 and 3.3 of fuzzy right tri-ideals in a semigroup.

Theorem 3.5. Let $\gamma \in F(S)$. Then $\gamma \in \mathcal{C}_{tir} \Leftrightarrow$ $\gamma \in F(S)$. Then $\gamma \in \mathcal{C}_{tir} \Leftrightarrow \gamma \circ \gamma \circ S \circ \gamma \subseteq \gamma$.

Proof: Let $z \in S$.

Then, $\gamma \circ \gamma \circ S \circ \gamma \subseteq \gamma$

$\Leftrightarrow \quad (\gamma \circ \gamma \circ S \circ \gamma)(z) \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min\{\gamma \circ \gamma(uv),\ S \circ \gamma(xy)\}] \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min(\gamma(u), \gamma(v), S(x), \gamma(y))] \leq \gamma(z)$

$\Leftrightarrow \quad \sup\limits_{z=uvxy} [\min(\gamma(u),\ \gamma(v), \gamma(x))] \leq \gamma(z)$

$\Leftrightarrow \gamma(uvxy) \geq \{\min[\gamma(u),\ \gamma(v),\ \gamma(x)]\}\ \forall\ u, v, x, y \in S$

$\Leftrightarrow \quad \gamma \in \mathcal{C}_{tir}$.

Finally equivalence between the Definitions 2.4 and 3.3 of fuzzy tri-ideals in a semigroup is provided and can be demonstrated in similar manner.

Theorem 3.6. If $\gamma \in F(S)$. Then, $\gamma \in \mathcal{C}_{ti} \Leftrightarrow \gamma \circ S \circ \gamma \circ \gamma \subseteq \gamma$ and $\gamma \circ \gamma \circ S \circ \gamma \subseteq \gamma$

Theorem 3.7. The classes $\mathcal{C}_{ss}, \mathcal{C}_l, \mathcal{C}_r, \mathcal{C}_i,\ \mathcal{C}_{in}, \mathcal{C}_b$ and $\mathcal{C}_q$ are projection closed.

Theorem 3.8. $\mathcal{C}_{til}$, the class of all fuzzy left tri-ideals of a semigroup S is projection closed.

Proof. Let $\gamma \in \mathcal{C}_{til}$. Therefore $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(z), \mu(w)\}\quad \forall\ x, y, z, w \in S$. To show $\mathcal{C}_{tir}$ is projection closed, we prove $\text{Rep}(\mu)(r) \in \mathcal{C}_{til}\ \ \forall\ r \in J.$ i.e. $\text{Rep}(\gamma)(r)(xyzw) \geq \min$ $\{\text{Rep}(\gamma)(r)(x), \text{Rep}(\gamma)(r)(z), \text{Rep}(\gamma)(r)(w)\}$ $\forall\ x, y, z, w \in S$. Let $x, y, z, w \in S$ and $r \in J$. Suppose $\min\{\text{Rep}(\gamma)(r)(x), \text{Rep}(\gamma)(r)(z), \text{Rep}(\gamma)(r)(w)\} = 1$. Therefore, $\text{Rep}(\gamma)(r)(x) = \text{Rep}(\gamma)(r)(z) = \text{Rep}(\gamma)(r)(w) = 1.$. This implies that $\gamma(x) > r,\ \gamma(z) > r$ and $\gamma(w) > r$. Thus $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(z), \gamma(w)\} > \mathrm{r}$ and hence $\text{Rep}(\gamma)(r)(xyzw) = 1.$

If $\min\{\mathrm{Rep}(x)(r)(x), \mathrm{Rep}(\gamma)(r)(z), \mathrm{Rep}(\gamma)(r)(w)\} = 0$, then the inequality holds trivially.

Theorem 3.9. $\boldsymbol{C}_{tir}$, the class of all fuzzy right tri-ideals of a semigroup S is projection closed.

Proof. Consider $\gamma \in \boldsymbol{C}_{til}$. Therefore $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(y), \mu(w)\}\ \forall\ x, y, z, w \in S$. To show $\boldsymbol{C}_{tir}$ is projection closed, we prove $\mathrm{Rep}(\gamma)(r) \in \boldsymbol{C}_{til}\ \forall\ r \in J.$ i.e. $\mathrm{Rep}(\gamma)(r)(xyzw) \geq \min$ $\{\mathrm{Rep}(\gamma)(r)(x), \mathrm{Rep}(\gamma)(r)(y), \mathrm{Rep}(\gamma)(r)(w)\}\ \forall\ x, y, z, w$. Let $x, y, z, w \in S$ and $r \in J$. Suppose $\min\{\mathrm{Rep}(\gamma)(r)(x), \mathrm{Rep}(\gamma)(r)(y), \mathrm{Rep}(\gamma)(r)(w)\} = 1$. Therefore, $\mathrm{Rep}(\gamma)(r)(x) = 1$ and *and* $\mathrm{Rep}(\gamma)(r)(y) = 1 = \mathrm{Rep}(\gamma)(r)(w)$. This implies that $\gamma(x) > r,\ \gamma(y) > r$ and $\gamma(w) > r$ Thus $\gamma(xyzw) \geq \min\{\gamma(x), \gamma(y), \gamma(w)\} > \mathrm{r}$ and hence $\mathrm{Rep}(\gamma)(r)(xyzw) = 1.$ If $\min\{\mathrm{Rep}(x)(r)(x), \mathrm{Rep}(\gamma)(r)(y), \mathrm{Rep}(\gamma)(r)(w)\} = 0$, then the inequality holds trivially.

Theorem 3.10. $\boldsymbol{C}_{ti}$, the class of all fuzzy tri-ideals of a semigroup S is projection closed.

4. Fuzzy Tri-ideals in Semirgroups and metatheorem

Theorem 4.1. In a semigroup S,

(i) Every left ideal (right ideal, two sided ideal, bi- ideal, interior ideal, quasi- ideal) of S is a tri-ideal of S.

(ii) Every fuzzy left(right) tri-ideal of a left (right) simple semigroup S is a fuzzy right (left) ideal of S.

(iii) Every fuzzy tri-ideal of left and right simple semigroup S is a fuzzy ideal of S.

Proof. We demonstrate (iii). Since both $\boldsymbol{C}_{ti}$ and $\boldsymbol{C}_{i}$ are closed under projection, therefore, $\boldsymbol{C}_{ti} \subseteq \boldsymbol{C}_{i} \Leftrightarrow C_{ti} \subseteq C_{i}$ by Proposition 2.4. By Corollary 3.9 of [9], every tri-ideal of a left and right simple semigroup S is also an ideal of S and P(S) C(S) under the isomorphism *Chi*, we get, $C_{ti} \subseteq C_{i}$. Hence $\boldsymbol{C}_{ti} \subseteq \boldsymbol{C}_{i}$

Since $\boldsymbol{C}_{l}\ (\boldsymbol{C}_{r}, \boldsymbol{C}_{i}, \boldsymbol{C}_{b}, \boldsymbol{C}_{in}, \boldsymbol{C}_{q}, \boldsymbol{C}_{til}, \boldsymbol{C}_{tir}, \boldsymbol{C}_{ti})$ are closed under projection by Theorem 3.7, 3.8 and 3.9. The remaining results follows in view Theorem 3.1, 4.2 and Corollary 3.8, 3.10, 4.1, 4.2 of [9], $P(S) \cong C(S)$ under the isomorphism *Chi* and Proposition 2.12.

Similarly, we can prove:

Theorem 4.2. Let S be a semigroup. Then,

(i) Let S be a regular semigroup. Then f is a fuzzy left (right) tri-ideal of S if and only if f is a fuzzy right (left) ideal of S.

(ii) Let S be a regular semigroup. Then f is a fuzzy tri-ideal of S if and only if f is a fuzzy two sided ideal of S.

Theorem 4.3 Let S be a semigroup and $h \in \boldsymbol{C}_{ss}$ such that Then $h \circ h = h$ Then, $h \in \boldsymbol{C}_{til}\ (\boldsymbol{C}_{tir}, \boldsymbol{C}_{ti})$ if and only if there exist $\gamma \in \boldsymbol{C}_{l}$ and $\alpha \in \boldsymbol{C}_{r}$ such that $a \circ \gamma \subseteq h \subseteq \gamma \cap \alpha$.

Proof. Define the classes D = $\{\chi_H \in C(S)$ where H is a subsemigroup of $S : A_1 B_1 \subseteq H \subseteq A_1 \cap B_1$ for some right ideal A₁ and left ideal B₁ of S} and D = $\{h \in \boldsymbol{C}_{ss} : \alpha \circ \gamma \subseteq h \subseteq \alpha \cap \gamma$ for some $\gamma \in \boldsymbol{C}_{l}$ and $\alpha \in \boldsymbol{C}_{r}\}$, where $C_r(C_l)$ is crisps classes of right (left) ideals of S. To prove D is projection closed, let $h \in$ D. Then, $h \in \boldsymbol{C}_{ss}$ such that $\alpha \circ \gamma \subseteq h \subseteq \alpha \cap \gamma$ for some $\alpha \in \boldsymbol{C}_{r}$ and $\gamma \in \boldsymbol{C}_{l}$. $\mathrm{Rep}(\alpha \circ \gamma)(r) \leq \mathrm{Rep}(h)(r) \leq \mathrm{Rep}(a \cap \gamma)(r)$ $\forall\ r \in J$ by Proposition 2.2. Now for $r \in J$, $\mathrm{Rep}(\alpha)(r) \circ \mathrm{Rep}(\gamma)(r) \leq \mathrm{Rep}(h)(r) \leq \mathrm{Rep}(\alpha)(r) \cap \mathrm{Rep}(\gamma)(r)$ by Proposition 2.1 and 2.3. Since $\boldsymbol{C}_{ss}$, $\boldsymbol{C}_{r}$ and $\boldsymbol{C}_{l}$ are projection closed, we have, $\mathrm{Rep}(\alpha)(r) \in \boldsymbol{C}_{r}$, $\mathrm{Rep}(\gamma)(r) \in \boldsymbol{C}_{l}$ and $\mathrm{Rep}(h)(r) \in \boldsymbol{C}_{ss}$ .Therefore, $\mathrm{Rep}(h)(r) \in \mathrm{D}\ \forall\ r \in J$. Hence D is Projection closed. Also $\boldsymbol{C}_{ti}$ is projection closed by Theorem 3.11. Therefore, by Proposition 2.4, $\mathrm{D} \subseteq \boldsymbol{C}_{ti} \Leftrightarrow D \subseteq C_{ti}$.

The later proposition follows as a subsemigroup K_1 of S with $K_1^2 = K_1$ is a left tri-ideal of S if $A_1B_1 \subseteq K_1 \subseteq A_1 \cap B_1$ for some right ideal A_1 and left ideal B_1 of S by Theorem 3.8 of [9] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

The following theorem extends Theorem 3.1 and 3.4 of [9] to the fuzzy context and can be proved in a similar manner.

Theorem 4.4. In a semigroup S and $\gamma \in \boldsymbol{C}_{ss}$. Then

(1) $\gamma \in \boldsymbol{C}_{til}$ if $S \circ S \circ \gamma \subseteq \gamma$. .

(2) $\gamma \in \boldsymbol{C}_{tir}$ if $\gamma \circ S \circ S \subseteq \gamma$.

(3) $\gamma \in \boldsymbol{C}_{til}$ if $\mathrm{S} \circ S \circ S \subseteq \gamma$.

(4) $\gamma \in \boldsymbol{C}_{ti}$ if $S \circ S \circ S \circ \gamma \subseteq \gamma$ and $\gamma \circ S \circ S \circ S \subseteq \gamma$.

Proof. (1). Define the classes $\boldsymbol{C}_A = \{S \circ \mathrm{S} \circ \gamma \subseteq \gamma \ \forall \gamma \in \boldsymbol{C}_{ss}\}$ and $C_A = \{\chi_A \in C(S),$ where A is a subsemigroup of $\mathrm{S}: SSA \subseteq A\}$. We first claim $\boldsymbol{C}_A$ is projection closed. Let $\gamma \in \boldsymbol{C}_A$. Therefore, $S \circ S \circ \gamma \subseteq \gamma \ \forall \gamma \in C_{ss}$ Now $\mathrm{Rep}(\gamma)(r) \geq \mathrm{Rep}(S \circ S \circ \gamma)(r) \forall r \in J$ by Proposition 2.2. Moreover, $\mathrm{Rep}(\gamma)(r) \geq \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\gamma)(r) = S \circ S$ by Proposition 2.3. Since $\boldsymbol{C}_{ss}$ is projection closed, therefore $\mathrm{Rep}(\gamma)(r) \in \boldsymbol{C}_{ss}$. Thus $\mathrm{Rep}(\gamma)(r) \in \boldsymbol{C}_A$ $\forall r \in J$ and hence $\boldsymbol{C}_A$ is closed under projection. Therefore $\boldsymbol{C}_A \subseteq \boldsymbol{C}_{til} \Leftrightarrow C_A \subseteq C_{til}$ by Proposition 2.4. As a subsemigroup A of a semigroup S is a left tri-ideal if SSA $\subseteq$ A by Theorem 3.1 of [9] and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_A \subseteq C_{til}$. Hence $\boldsymbol{C}_A \subseteq \boldsymbol{C}_{til}$.

Theorem 4.5. A semigroup that is commutative idempotent is regular if and only if $\gamma \circ S \circ \gamma \circ \gamma = \gamma \ \forall \gamma \in \boldsymbol{C}_{ti}$.

Proof. Let $(S, \circ)$ be a semigroup. Consider the algebra- $(F(S), \inf, \sup, \circ)$. let $L(a) = aSaa$ and $M(a) = a$ be expression over the variable set {a} and operations set $\{\inf, \sup, \circ\}$. Since the class $\boldsymbol{C}_{ti}$ of fuzzy tri-ideals is projection closed, therefore, $L(\gamma) = M(\gamma) \ \forall \ \gamma \in \boldsymbol{C}_{ti}$ if and only if $L(\gamma) = M(\gamma) \ \forall \ \gamma \in C_{ti}$ by metatheorem.

Since a semigroup S that is commutative idempotent is regular

$\Leftrightarrow ASAA = A$ for any tri-ideal A of *S*.

$\Leftrightarrow \chi_{ASAA} = \chi_A$ for any tri-ideal A of *S* since $P(S) \cong C(S)$ under the isomorphism *Chi* $\Leftrightarrow \chi_A \chi_S \chi_A \chi_A = \chi_A$ for any tri-ideal A of *S* since *Chi* commutes with the finite intersection and product of sets in a semigroup S.

$\Leftrightarrow \chi_A \mathrm{S} \chi_A \chi_A = \chi_A$ for any tri-ideal A of *S*

$\Leftrightarrow L(\gamma) = M(\gamma) \ \forall \gamma \in C_{ti}$

$\Leftrightarrow L(\gamma) = M(\gamma) \ \forall \gamma \in \boldsymbol{C}_{ti}$.

$\Leftrightarrow \gamma \circ S \circ \gamma \circ \gamma = \gamma \ \forall \gamma \in \boldsymbol{C}_{ti}$.

The subsequent theorem can be derived in a similar manner.

Theorem 4.6. A commutative idempotent semigroup is regular if and only if $\gamma \circ S \circ \gamma \circ \gamma = \gamma \ \forall \gamma \in \boldsymbol{C}_{til}$.

Theorem 4.7. In a semigroup S,

(1) If $\beta \in \boldsymbol{C}_r$ and $\gamma \in \boldsymbol{C}_i$, then $\beta \circ \gamma \in \boldsymbol{C}_{ti}$.

(2) If $\beta \in \boldsymbol{C}_r$ and $\gamma \in \boldsymbol{C}_i$, then $\beta \cap \gamma \in \boldsymbol{C}_{ti}$.

Proof. **(1)**. Define the classes $C_{r,l} = \{\alpha_1 \circ \alpha_2 : \alpha_1 \in C_r, \alpha_2 \in C_l\}$ and $\boldsymbol{C}_{r,l} = \{\beta \circ \gamma : \alpha \in \boldsymbol{C}_r, \gamma \in \boldsymbol{C}_l\}$, where $\boldsymbol{C}_r$ ($\boldsymbol{C}_l$) is crisp classes of right (left) ideals of S. To show that $\boldsymbol{C}_{r,l}$ is projection closed, let $\beta \cap \gamma \in$ $\boldsymbol{C}_{r,l}$. Now, for all $\beta \in \boldsymbol{C}_r$ and $\gamma \in \boldsymbol{C}_l$, $\mathrm{Rep}(\beta \circ \gamma)(r) = \mathrm{Rep}(\beta)(r) \circ \mathrm{Rep}(\gamma)(r) \ \forall r \in J$ by Proposition 2.3. $\mathrm{Rep}(\beta)(r) \in \boldsymbol{C}_r$ and $\mathrm{Rep}(\gamma)(r) \in \boldsymbol{C}_l$ since both $\boldsymbol{C}_r$ and $\boldsymbol{C}_l$ are projection closed. Thus $\mathrm{Rep}(\beta \circ \gamma)(r) \in \boldsymbol{C}_{r,1} \ \forall \ r \in J$. Hence, the classes $\boldsymbol{C}_{r,l}$ is projection closed. Since $\boldsymbol{C}_{ti}$ is also projection closed by Theorem 3.9, therefore,

$\boldsymbol{C}_{r,l} \subseteq \boldsymbol{C}_{ti} \Leftrightarrow C_{r,l} \subseteq C_{ti}$ by Proposition 2.4. Since the product of a right ideal and a left ideal of a semigroup is a tri-ideal by Theorem 3.1 of [9] and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{r,l} \subseteq C_{ti}$. Hence, $\boldsymbol{C}_{r,l} \subseteq \boldsymbol{C}_{ti}$.

The remaining results extends Theorem 3.1 of [9] to fuzzy setting and can be proved in a similar manner.

Follows in view Theorem 3.1, 4.2 and Corollary 3.8, 3.10, 4.1, 4.2 of [9].

Theorem 4.8. In a semigroup S,

(1) if $\beta \in \boldsymbol{C}_{ti}$ and $\gamma \in \boldsymbol{C}_{in}$ $(\boldsymbol{C}_i)$, then $\beta \cap \gamma \in \boldsymbol{C}_{ti}$.

(2) if $\beta \in \boldsymbol{C}_{tir}$ and $\gamma \in \boldsymbol{C}_{til}$, then $\beta \cap \gamma \in \boldsymbol{C}_{ti}$.

(3) if $\beta \in \boldsymbol{C}_{ti}$ and $\gamma \in \boldsymbol{C}_{in}$ $(\boldsymbol{C}_i,\ \boldsymbol{C}_b)$,
Then $\beta \cap \gamma \in \boldsymbol{C}_{ti}$.

(4) $\beta \in \boldsymbol{C}_b$ and $\gamma \in \boldsymbol{C}_{in}$, then $\beta \cap \gamma \in \boldsymbol{C}_{ti}$

Proof. (3). Define the classes in S as $C_{ti,\ \text{in}} = \{\alpha_1 \cap \alpha_2 : \alpha_1 \in C_{ti},\ \alpha_2 \in C_{in}\}$ and $\boldsymbol{C}_{ti,\ \text{in}} = \{\beta \cap \gamma : \alpha \in \boldsymbol{C}_{ti},\ \gamma \in \boldsymbol{C}_{in}\}$, where C_{ti} and C_{in} are crisp classes of tri-ideals and interior ideals of S. To show that $\boldsymbol{C}_{ti,\ \text{in}}$ is projection closed, let $\beta \cap \gamma \in \boldsymbol{C}_{ti,\ \text{in}}$. Now, for all $\beta \in \boldsymbol{C}_{ti}$ and $\gamma \in \boldsymbol{C}_{in}$, $\text{Rep}(\beta \cap \gamma)(r) = \text{Rep}(\beta)(r) \cap \text{Rep}(\gamma)(r) \quad \forall r \in J$ by Proposition 2.3. $\text{Rep}(\beta)(r) \in \boldsymbol{C}_{ti}$ and $\text{Rep}(\gamma)(r) \in \boldsymbol{C}_{in}$ since both $\boldsymbol{C}_{ti}$ and $\boldsymbol{C}_{in}$ are projection closed. Thus $\text{Rep}(\beta \cap \gamma)(r) \in \boldsymbol{C}_{r,1}\ \forall r \in J$. Hence, the $\boldsymbol{C}_{ti,\ \text{in}}$ is projection closed. Since $\boldsymbol{C}_{ti}$ is also projection closed, therefore, $\boldsymbol{C}_{ti,\ \text{in}} \subseteq \boldsymbol{C}_{ti} \Leftrightarrow C_{ti,\ \text{in}} \subseteq C_{ti}$ by Proposition 2.4. Since the intersection of a tri-ideal and an interior ideal of a semigroup S is also a tri-ideal by Corollary 3.10 of [9] and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{ti,\ \text{in}} \subseteq C_{ti}$. Hence, $\boldsymbol{C}_{ti,\ \text{in}} \subseteq \boldsymbol{C}_{ti}$.

The remaining results extends Theorem 3.13, 3.14, 3.15, Corollary 3.5, 3.13 of [9] to fuzzy setting and can be proved in a similar manner.

References

[1] R. A. Good and D. R. Hughes, Associative groups for a semigroup, Bull. Amer. Math. Soc., 58 (1952) 624--625.

[2] T Head, A metatheorem for deriving fuzzy theorems from crisp versions, FSS, 73(1995) 349-358.

[3] K Iseki, Quasi-ideals in semirings without zero, Proc. Japan Academy, 34(2) (1958) 79-84.

[4] A Jain, Tom Head's join structure of fuzzy subgroups, Fuzzy Sets and Systems, 125(2002) 191-200.

[5] Nobuaki Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5 (1981) 203-215.

[6] Nobuaki Kuroki, On fuzzy interior ideals in semigroups, J. Fuzzy Math.}, 3 (2) (1988) 436-447.

[7] S. Lajos and F.A. Szasz, On the bi-ideals in associative ring, Proc. Japan Acad. 46(1970) 505-507.

[8] M.M.K. Rao, Tri-ideals of semirings, Bull. Int. Math. Virtual Inst, 10(1)(2020), 145-155.

[9] M.M.K. Rao, Tri-ideals of semigroups, Bull. Int. Math. Virtual Inst, 14(2) (2024) 213-2233.

[10] Anand Swaroop Prajapati and Ravi Srivatava, Metatheorem and fuzzy quasi-ideals in semigroups, The Journal of Fuzzy Mathematics, 14 (4) (2006) 851-865.

[11] Ravi Srivastava and Arvind, Generalization of Fuzzy ideals of Semirings and Metatheorem, Annals of Fuzzy Mathematics and Informatics, 27 (3) (2024), 245-255.

[12] G Szasz, Interior ideals in semigroups, Notes on Semigroups IV, Dept. Math. K. Marx. Univ. Economics, (1977) 1-7.

[13] L A Zadeh, Fuzzy Sets, Inform. and Control}, 8(1965) 338-353.