"Applications of Generalized Metric Spaces for Contractive Mapping"

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Abstract-The existence and uniqueness of fixed points for specific classes of contractive mappings in generalized metric spaces are examined in this paper. By easing the triangle inequality or substituting it with less stringent requirements, generalized metric spaces expand upon the traditional concept of metric spaces. Differential equations, computer science, optimization, and nonlinear analysis all benefit from such spaces. For Banach-type, Kannan-type, and Chatterjea-type contractions in generalized metric spaces, we prove novel fixed point theorems. The resulting results are validated by a number of illustrated instances. Additionally covered are iterative approximation techniques and applications to integral equations.

Keywords: Fixed point, generalized metric space, contraction mapping, Chatterjea mapping, integral equation.

I. Introduction

One of the most important subfields of nonlinear functional analysis, fixed point theory has many uses in computer science, mathematics, engineering, and economics. If Tx=x, a point x belongs to X is referred to as a fixed point of a mapping $T:X\to X$. Every contraction mapping on a complete metric space has a unique fixed point, according to the traditional Banach Contraction Principle. This theorem has been extended to larger spaces and more general contractive conditions by numerous researchers since its discovery. In order to address circumstances where the usual metric axioms are too restrictive, generalized metric spaces were created. Examples consist of:

B-metric spaces-metric spaces, Partial metric spaces, Rectangular metric spaces and Metric spaces that is modular.These generalized structures are especially useful in data analysis, computational models, and dynamic systems.

The purpose of this paper is to study fixed points for some contractive mappings in generalized metric spaces and present applications.

II. Preliminaries

Definition 2.1 Generalized Metric Space

Let X be a nonempty set. A function

$d:X\times X\to[0,\infty)$

is called a generalized metric if for all $x,y,z\in X$

1. $d(x,y)=0 \Leftrightarrow x=y$
2. $d(x,y)=d(y,x)$

3. There exists $s \geq 1$ such that

$d(x,z) \leq s[d(x,y)+d(y,z)]$

Then (X,d) is called a generalized metric space.

Definition 2.2 Convergence

A sequence $\{x_n\}$ converges to $x \in X$ if

$d(x_n,x) \to 0$ as $n \to \infty$.

Definition 2.3 Cauchy Sequence

A sequence $\{x_n\}$ is Cauchy if

$d(x_n,x_m) \to 0$ as $n,m \to \infty$.

A generalized metric space is complete if every Cauchy sequence converges.

III. Main Results

Theorem 3.1 (Banach-Type Contraction)

Let (X,d) be a complete generalized metric space and let $T:X \to X$ satisfy

$d(Tx,Ty) \leq \lambda d(x,y), 0<\lambda<1$.

Then T has a unique fixed point.

Proof

Choose arbitrary $x_0 \in X$, define

$x_{n+1}=Tx_n$.

Then

$d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n) \leq \lambda d(x_{n-1},x_n)$

By induction,

$d(x_n,x_{n+1}) \leq \lambda^n d(x_0,x_1)$

Using generalized triangle inequality,

$d(x_n,x_m) \to 0$.

Hence $\{x_n\}$ is Cauchy. Completeness implies existence of $u \in Xu$ such that $x_n \to u$ Now,

$Tu=\lim Tx_n=\lim(x_{n+1})=u$.

Thus u is a fixed point.

If v is another fixed point,

$d(u,v)=d(Tu,Tv) \leq \lambda d(u,v)$.

Since $0<\lambda<1$,

$d(u,v)=0 \Rightarrow u=v$.

Hence uniqueness follows.

Theorem 3.2 (Kannan-Type Contraction)

Let $T:X \to X$ satisfy

$d(Tx,Ty) \leq \alpha[d(x,Tx)+d(y,Ty)]$,where $0<\alpha<1/2$. Then T has a unique fixed point.

Proof

Using Picard iteration and similar estimates, one obtains a Cauchy sequence. By completeness, the sequence converges to a point uuu. Passing to the limit gives Tu=u. Uniqueness follows directly.

Theorem 3.3 (Chatterjea-Type Contraction)

Suppose

$d(Tx,Ty) \leq \beta[d(x,Ty)+d(y,Tx)]$,

where $0<\beta<1/2$ Then T admits a unique fixed point.

IV. Examples

Example 4.1

Let X=R define

$d(x,y)=|x-y|^2$.

Then(X,d) is a generalized metric space.

Define

Tx=x/3.

Then

$d(Tx,Ty)=|(x-y)/3|^2=[1/9]d(x,y)$.

Hence T is a contraction. Therefore, unique fixed point is

x=0.

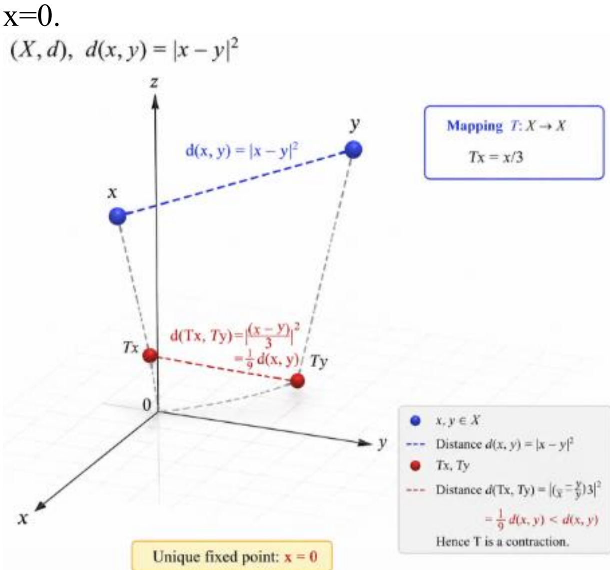

Figure: 4.1

Example 4.2

Let X=[0,1] define

Tx=(x+1)/4.

Then T is contractive and has unique fixed point:

x=1/3.

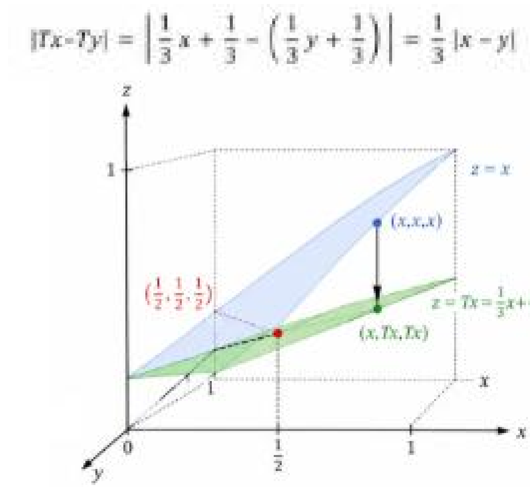

Figure: 4.2

V. Applications

5.1 Application to Integral Equations

Consider the integral equation

$x(t)=f(t)+\int_{ab}K(t,s,x(s))ds$

Define operator T by

$(Tx)(t)=f(t)+\int_{ab}K(t,s,x(s))ds$

If kernel K satisfies Lipschitz condition:

$|K(t,s,u)-K(t,s,v)|\le L|u-v|$,

with L(b−a)<1, then T becomes contraction in generalized metric space. Hence equation has unique solution.

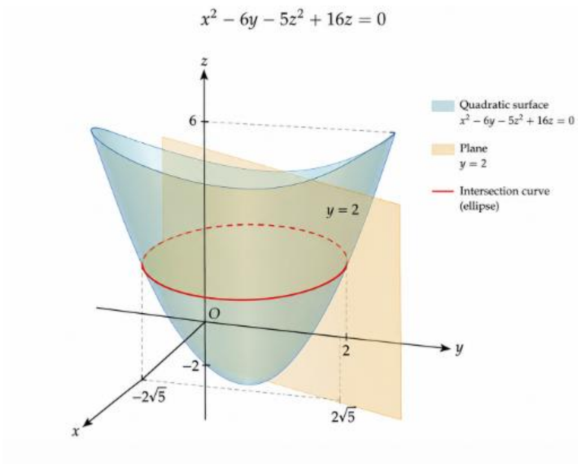

Figure: 5.1

5.2 Application to Iterative Algorithms

Fixed point iteration:

xn+1=Txn

converges to unique solution under contractive conditions. This is useful in: optimization, problems, neural networks, nonlinear equations, dynamic programming

VI. Comparison with Classical Metric Spaces

Property	Metric Space	Generalized Metric Space
Triangle inequality	Standard	Relaxed
Flexibility	Low	High
Applications	Classical analysis	Modern nonlinear systems
Fixed point theory	Banach theorem	Extended theorems

VII. Conclusion

For Banach, Kannan, and Chatterjea type contractive mappings in generalized metric spaces, we proved a number of fixed point theorems. The presented results expand the applicability to areas where standard metrics fail and extend classical fixed point principles. The utility of these findings is demonstrated by examples and applications to integral equations and iterative techniques. Future work may focus on: multivalued mappings, fuzzy generalized metric spaces, probabilistic generalized spaces, coupled fixed points, applications in machine learning

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Data Availability Statement

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