Optimal Pollution Control for Persistence Of Biological Species

I.Laxmi Gayatri , Lecture in Mathematics,

Govt College (A), Rajumandry

, Andhra Pradesh, India (Bharat)

ilggovt@gmail.com

Abstract: In this paper, a Mathematical model representing the dynamics of biological species in a polluted environment is described and analyzed. The dynamical behavior of the model is investigated and examine when an exogenous rate of toxicant enter into ecosystem. This model contains state variables representing population biomass, the concentration of toxicant in organism and the concentration of the toxicant in the environment. This Modeling analysis discusses system level effects of toxicants on population (or) organism when exposure is via both environmental and food chain path ways (or closer –response parameters). The proposed model is consider as dynamic optimal control) problem with state constraints. By applying control theory to proposed model to control the pollution (or toxicant influx) such that pollution level (or toxicant) never exceeds a specified allowable level and that species population can survive and reach at desired level in given habitat to maintain ecological balance. Some numerical computations are also given:

AMSClassification:37-02,37D99,34L15,34L20,37C75,49K30,49K15,49J35.

Introduction: The effects of toxicants on ecological communities have become problems of major environmental concerns in the recent decades. With the rapid development of modern industry and agriculture, a great quantity of toxicant and contaminate enter into ecosystem one after the other. These pollutants threat the survival of the exposed population. Therefore it is important to study the effect of toxicant on population and steps should take to control the toxicant level for persistence or conservation of a population or community. The problem of estimating qualitatively the effects of toxicant on a population by mathematical models begin on the early 1980's. Mathematical Modelling in dealing with such ecotoxicological problems started with the studies of Hallam and Clark (1982), Hallam (1983), Hallam and DeLuna (1984), Freedman and Shukla (1991) and others. These experimental studies have been conducted to study the effect of a single toxicant on both terrestrial [Gased S.G.(1981), Reinor (1981)] and acquatic [Patin (1982), Singh (1991)] ecosystems. There are several; papers investigated the models of the persistence and extinction of a population in a polluted environment. In 1989 Ma etal (1988), Hallam and Ma (1989) and Mazhien (1990) studied the sufficient condition on persistence or extinction of a population and obtain a threshold between two in most situation. In the work Liu Humpes and Mazhien (1990), the threshold results are expressed in terms of relationship involving the population intrinsic growth rates, dose – response parameters and interaction rates. In other works like Srinivasu (2001) etal, examine to find some sufficient condition depending an parameters of the model and effort has been made to find out a clean up policy implemented at source level of pollutant in order to conserve a population.It can be found that there are several literatures use the optimal control theory for ecotoxicological models. The use of optimization techniques in ecology has been widely advocated by Watt ([1], chapter [13]). An optimization technique for dynamic system is Pontryagin's maximum principle; it is a modern form of the classical calculus of variations. Literatures like Otto (1980), Nelson (1970), Jensons (1982), Naeem (1994), Caputo (1995), Keeler (1972), Forster (1972) and Sakewa (1978) used the optimal control theory for mathematical models of ecology. Fleming (1974), Gruer (1996) discussed the problem by using the optimal control theory. Keeping in mind these studies, in this work a mathematical model is proposed to study the impact of toxicant influx released from the source influencing growth of Biological species. By applying the optimal control theory to this model to control the emission rate of toxicant at the source so that it leads to persistence of population and maintain ecological balance. This work is described as follows. In the first section a Mathematical model representing ecotoxicological is described. In second section we discuss and analyze the stability nature (both local and global) of interior equilibrium points of the considered system . In third section dynamic optimal control theory is used to control the toxicant influx at the source level. In the fourth section numerical simulation is given to describe the nature of dynamics for the given conditions.

Mathematical Modeling: The state variables of the model are x = x (t), the population at a time t, $C_o = C_o(t)$, the concentration of toxicant present in the organism at time t and $C_E = C_E(t)$, the concentration of toxicant in the environment at time t. The dynamics of the population are assumed to be governed by a logistic equation in the absence of pollution and the growth rate of population decreases due to uptake of toxicant. The rate of change of concentration of toxicant in the organism increases and proportional to concentration due to natural degradation process. Now the concentration of toxicant level in an environment decreases due to natural degradation and increases with exogeneous rate of toxicant input into environment which releases from source level and also due to death of species or wastages released from species which are effected by pollution. All these expressions are linear. Therefore the following model describes the dynamics of single – species in a pollutant environment

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\alpha_1 C_o x$$

$$\frac{dC_o}{dt} = a_1 C_E - \delta_1 C_o + \theta_1 \qquad (2.1)$$

$$\frac{dC_E}{dt} = -hC_E + \alpha_2 C_o + u(t)$$

Where r>o is the intrinsic growth rate of the population in the environment without toxicant; k>o is carrying capacity of environment that supports the growth of organism such the < x (o) < x (t) < k; $\alpha_1 > 0$ is the decreasing rate of toxicant; $a_1 > 0$ represents the organism's net uptake of toxicant for the environment, $\delta_1 > 0$

represents egestion and depuration rates of toxicant in the environment, $\theta_1 = \frac{d_1\theta\beta}{a_1}$, where $a_1 > 0$, environment toxicant uptake rates per unit mass organism; $d_1 > 0$ the uptake rate of toxicant food per unit mass organism, $\theta > 0$ the concentration of toxicant in the resource, β the positive constant is the average rate of food intake per unit mass organism. The positive constant h represents the loss rate of toxicant from the environment including processes such as biological transformation chemical hydrolysis, volatilization, microbial degradation and photosynthetic degradation; $\alpha_2 > 0$ increase in the concentration of toxicant in the environment coming from the egestion of the total population. The function u(t) represents the exogenous rate of input of toxicant into the environment released from the source. Here u (t) is bounded such that $0 \le u_0 \le u(t) < u_1 < \infty$, for constants u_0 and u_1 and for all t in [o, ∞).The model (2.1) satisfy the following conditions :.i)$x(0) > 0$, $C_0(0) = 0$, $C_E(0) > 0$;ii)$0 < C_E(0) < C_E(t)' < C_E^*$, where C_E^* is the maximum level of concentration of toxicant in the environment such that environment can absorb.iii)$0 < C_0(0) < C_0(t) < C_0^*$ where C_0^* is constant is the maximum level of concentration of toxicant present in an organism, at which species will survive but can't reproduce.Therefore the model (2.1)ith above initial conditions is known as dynamics of single – species influenced by toxicant influx. Here we consider u(t) equal to u as a constant. Now we investigate the dynamics of the exploited system (2.1).

Steady States: The possible steady states of the dynamical system (2.1) can be obtained by equating the equations of (2.1) to zero. Then the positive equilibrium point of the considered system is $P(\bar{x}, \overline{C_0}, \overline{C_E})$ .where $\bar{x} = k\left(1 - \frac{\alpha_1}{r}\overline{C_0}\right), \overline{C_E} = \frac{\alpha_2 \overline{C_0} + u}{h}, and\ \overline{C_0} = \frac{\theta_1 h + a_1 u}{\delta_1 h - \alpha_2 a_1}$.

It can be observe that the $\overline{C_0}$ is positive for $\delta_1 h - \alpha_2 a_1 > 0$. Since δ_1 and h are natural degradation constants and they are high in value in generally hence $\overline{C_0}$ is positive. Also we have $0 < x(0) < x(t) < \overline{x}$ $< k$ and it can be observe that $\overline{C_E} > 0$. Now we discuss the stability nature of the steady state $P\left(\overline{x},\overline{C_0},\overline{C_E}\right)$ for the system (2.1) by method of variational matrix. The Jacobian or variational matrix of the considered system is given by

$$V\left(x,C_0,C_E\right)=\begin{bmatrix} r-\dfrac{2rx}{k}-\alpha_1 C_0 & -\alpha_1 x & 0 \\ 0 & -\delta_1 & a_1 \\ 0 & \alpha_2 & -h \end{bmatrix} \quad (2.2).$$

Evaluating (2.2) at $P\left(\overline{x},\overline{C_0},\overline{C_E}\right)$ we get

$$V\left(\overline{x},\overline{C_0},\overline{C_E}\right)=\begin{bmatrix} -r+\alpha_1 C_0 & -\alpha_1 \overline{x} & 0 \\ 0 & -\delta_1 & a_1 \\ 0 & \alpha_2 & -h \end{bmatrix} \quad (2.3).$$

Thus the characteristic equation of (2.3) is given by

$$\left(-r+\alpha_1\overline{C_0}-\lambda\right)\left[\left(\delta_1+\lambda\right)\left(h+\lambda\right)+a_1\alpha_2\right]=0 \quad (2.4).$$

Therefore the roots of the above equation are

$\lambda_1 = \left(-r + \alpha_1 \overline{C_0}\right) and \left(\delta_1 + \lambda\right)\left(h + \lambda\right) + a_1 \alpha_2 = 0$ .Here $\lambda_1 < 0 \; if \; C_0 < \dfrac{r}{\alpha_1}$,

[which is obvious], since the concentration of toxicant present in species be smaller value than the ratio of intrinsic growth rate and death rate, otherwise the species can't survive in a given habitat. In the second equation the remaining two roots are negative $(\lambda_2 < 0 \; and \; \lambda_3 < 0)$ is obvious. Therefore it can be concluded that the roots of (2.4) are negative, hence the point P is stable. With the following proposition we prove the global stability of the system (2.1).

Proposotion 2.1 : Let the following inequalities hold than (i) $\left(\dfrac{\overline{C_0}}{\overline{C_0}} \dfrac{\overline{x}}{k} - \dfrac{x}{k}\right) > 0$ and

(ii) $\left[a_1 \left(C_0 - \overline{C_0}\right) - C_E - \overline{C_E} \dfrac{C_0}{C_0}\right] > 0$ then $P\left(\overline{x}, \overline{C_0}, \overline{C_E}\right)$ is globally stable.

Proof: Consider the Liapunov function

$$V\left(x, C_0, C_E\right) = \left[\left(x - \overline{x}\right) - \overline{x} \log \frac{x}{x}\right] + \frac{1}{2}\left(C_0 - \overline{C_0}\right)^2 + \frac{1}{2}\left(C_E - \overline{C_E}\right)^2$$

(2.5)The above function satisfy the following condition(i) $V\left(\overline{x}, \overline{C_0}, \overline{C_E}\right) = 0$

(ii) $V\left(x, C_0, C_E\right)$ is positive definite.Now we prove that $\frac{dV}{dt} = \dot{V}$ is negative definite.

$$\dot{V} = \left(\frac{x-\bar{x}}{x}\right)\left[rx\left(1-\frac{x}{k}\right)-\alpha_1 x C_0\right]+\left(C_0-\overline{C_0}\right)\left[a_1 C_E-\delta_1 C_0+\theta_1\right]+$$

$$\left(C_E-\overline{C_E}\right)\left[-hC_E+\alpha_2 C_0+u\right]$$

$$\dot{V} = r\left(x-\bar{x}\right)\left(1-\frac{C_0}{\overline{\overline{C_0}}}\right)+r\left(x-\bar{x}\right)\left[\frac{C_0}{\overline{\overline{C_0}}}\frac{\bar{x}}{k}-\frac{x}{k}\right]-\delta_1\left(C_0-\overline{C_0}\right)^2$$

$$+\left(C_E-\overline{C_E}\right)\left[\left(1-\frac{C_0}{\overline{\overline{C_0}}}\right)u\right]+\left(C_E-\overline{C_E}\right)\left[a_1\left(C_0-\overline{C_0}\right)-C_E+\overline{C_E}\frac{C_0}{\overline{\overline{C_0}}}\right]$$

using inequalities (i) and (ii) in the statement it implies dV/dt < 0. Hence the system is (2.1) is globally stable and does not possess any periodic solution.

Dynamic Optimal Control: The purpose of this contribution is to illustrate the use of optimal control theory to obtain optimal strategies for the control of pollution (or toxicant influx at source). In this work we apply the theory of optimal control in Cesari [1983]. And we apply Lagargene's method of the Cesari [1983]. Therefore for the system (2.1) the optimal control problem presented in the following form:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\alpha_1 C_0 x;$$

$$\frac{dC_0}{dt} = a_1 C_E+\theta_1-\delta_1 C_0;$$

$$\frac{dC_E}{dt} = -hC_E+u(t)-\alpha_1 C_0;$$

performance Index J = I(u) = $\int_{t_0}^{t_f} u(t)\,dt$ (2.5). with intial conditions.The optimal control problem is to find an admissible control function u(t) which drives the system from the observed state at intial state to P(interior equilibrium point). And such that the performance Index J is minimized. For our model we apply Lagarange's optimal control theorem to minimize the index function J. The Hamiltonian function for this problem is given by

$$H(x, C_0, C_E, u, \lambda_1, \lambda_2, \lambda_3, \lambda_0) = \lambda_0 u + \lambda_1\left(rx\left(1-\frac{x}{k}\right)-\alpha_1 C_0 x\right)$$

$$+\lambda_2\left(a_1 C_E - \delta_1 C_0 + \theta_1\right)$$
$$+\lambda_3\left(-hC_E + \alpha_2 C_0 + u(t)\right) \quad (2.6)$$

For optimal control u = u(t) and trajectory (x(t), C_0(t), C_E (t)) for to ≤ t ≤ t^*, the adjoint variables λ_0, λ_1, λ_2, λ_3 not all zeros which satisfy λ_0 = constant > 0 and

$$\dot{\lambda}_1 = \frac{-\partial H}{\partial x} = \lambda_1\left(r - \frac{2rx}{k} - \alpha_1 C_0\right) \quad (2.7)$$

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial C_0} = \lambda_1(-\alpha_1 x) + \lambda_2(-\delta_1) + \lambda_3(\alpha_2) \qquad (2.8)$$

$$\dot{\lambda}_3 = \frac{-\partial H}{\partial C_E} = \lambda_2(a_1) + \lambda_3(-h)$$

(2.9)Further more

H (x^*(t), $C_0{}^*$(t), $C_E{}^*$(t), u^*(t), $\overline{\lambda(t)}$) = 0 (2.10) where $\overline{\lambda(t)}$ = (λ_0, λ_1, λ_2, λ_3) and along an optimal trajectory the optimal control u(t) minimizes H (x(t), C_0(t), C_E (t), u(t), λ(t)) with respect to all admissible controls. Hence

$H_u > 0$ implies $u^*(t) = 0$ (2.11) and $H_u < 0$ implies $u^*(t) = u_{max}$ (2.12)

If on the subinterval [t_1, t_2] $H_u = \lambda_0 + \lambda_3 = 0$ implies $\lambda_3 = -\lambda_0$ (2.13).Since u appears linearly in H we have a singular control.Differentiating (2.6) in totally, then $DH_u = 0 \Rightarrow \dot{\lambda}_3 = 0$ this implies

$$\lambda_2 = -\frac{h}{a_1}\lambda_0$$

(2.14). Using (2.13) in (2.14) we get

$$\lambda_2 = -\frac{h}{a_1}\lambda_0$$

(2.16).

We are interested in state which satisfy x(t) > 0, C_E (t) > 0 similarly $0 = D^2 H_u$ this implies

$$\lambda_1 \alpha_1 x = -\lambda_0 \left(\alpha_2 - \frac{h}{a_1} \right) \qquad (2.16)$$

Again $0 = D^3 H_u$ this implies $\lambda_1 \alpha_1 x\left(\frac{rx}{k}\right) = 0$ (2.17)

Hence $\lambda_1 = 0$ which leads to $\lambda_0 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$.Which is a contradiction. By Langarge's optimal control theory not all adjoint variables are zero. Hence 'u' is not singular.Therefore u takes the values between u = 0 and $u = u_{max}$. Hence the optimal solution of system (2.1) lies between u = 0 and $u = u_{max}$.

Conservation through control on toxicant To analyze the qualitative nature of the function u(t), which represents the rate of toxicant entered into the environment, let us assume that the rate of toxicant entered into the environment, produced at the source is proportional to consumption of the nature c(t). This consumption nature c (t) can be defined as $\frac{dc}{dt} = -\delta e^{-t}$ with initial condition c (0) = μ, whose solution is in the form of $c(t) = \delta_1\left[\exp(-t) + \mu\right]$, where exp(-t) represents the decrease in rate of nature degradation (due to deforestation etc) and $\delta_1 > 0$ represents decrease rate in assimilation nature due to lack of conservation policy . Suppose certain amount of toxicant concentration at the source removed is proportional to the amount of the pollutant produced at this source, then the dynamics of the toxicant (Pollutant) produced at the source is formulated as $\frac{du}{dt} = \delta_1\left(\exp(-t) + \mu\right) - \eta\, u$,with initial Condition $u(0) = u_0$ and $\eta > 0$ represents clean up rate at the source, where u_0 is a very small (negligible) quantity of emission released from source initially. Then the solution of the above differential equation is

$$u(t) = \frac{\delta_1}{\eta - 1}\left[e^{-t} - e^{-\eta t}\right] + \frac{\delta_1 \mu}{\eta}\left[1 - e^{-\eta t}\right] + u_0\ e^{-\eta t} \qquad (8)$$

This solution u(t) is

bounded and $0 < u_0 \leq u(t) \leq u_1 < \infty$. where $u_1 = \dfrac{\delta_1 \mu}{\eta}$. Equation (8) gives the rate of toxicant emission or discharge entering into environment for conservation (or persistent) of population.

Numerical Simulation: In this section we verify important results developed in the previous section through an illustrative example. In the following table gives the values of the parameters of system (2.1).

r	k	α_1	a_1	δ_1	θ_1	h	α_2	δ	η , μ
0.6	25	0.28	0.22	1.3	0.0036	2.2	0.23	0.01	1.2,0.5

Table 2.1

As u(t), the exogenous input of influx, rate increases we can observe that the concentration of toxicant in organism i.e $C_0(t)$, and in environment $C_E(t)$ increases and single species population decreases. As long as u(t) value is between $u_{min} = 0$ and $u_{max} = 4.1$ the solution of the system reaches equilibrium point in a

finite time. Graph 2.1

Conclusion: In this paper we have described a model representing the dynamics of effects of toxin introduced into the environment of single species. We investigate the effect of toxicant on both population and an environment. It observe that, more toxicant introduced into environment increase the concentration level of toxicant in organism and in environment and decrease the species population. Therefore to control the toxicant emission, we apply Lagargen's optimal control in Cesari (1983) to control on toxicant influx at source level for persistence of species population. From dynamical optimal control, analysis it observe that the system (2.1) posses optimal solution ($x^*(t)$, $C_0^*(t)$, $C_E^*(t)$) as long u(t) takes the values between u $=u_{imin}$and $u = u_{Max}$ in a finite time. From numerical analysis it examine the solution of the system under given conditions reaches the equilibrium point in a finite line when there is a control on exogenous input u(t). Therefore from the analysis of qualitative structure and numerical simulation of the system 2.1, it can be concluded that an appropriate level of species population can maintained by controlling the influx of toxicant at source as well as by clearing the environment using some removal mechanism and these maintains ecological balance.

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$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha_1 C_o x$$

$$\frac{dC_o}{dt} = a_1 C_E - \delta_1 C_o + \theta_1$$

) is

$$\lambda_1 = \left(-r + \alpha_1 \overline{C_0}\right) and \left(\delta_1 + \lambda\right)\left(h + \lambda\right) + a_1 \alpha_2 = 0$$

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$$r\left(x - \bar{x}\right)\left(1 - \frac{C_0}{\overline{C_0}}\right) + r\left(x - \bar{x}\right)\left(-\frac{x}{k} + \frac{C_0}{\overline{C_0}}\frac{\bar{x}}{k}\right)$$

$$+\left(C_0 - \overline{C_0}\right) a_1 \left(C_E - \overline{C_E}\right) - \delta_1 \left(C_0 - \overline{C_0}\right)^2 + \left(C_E - \overline{C_E}\right)\left[\left(1 - \frac{C_0}{\overline{C_0}}\right)u\right]$$

$$+\left(C_E - \overline{C_E}\right)\ \left(-C_E + \overline{C_E}\frac{C_0}{\overline{C_0}}\right)$$

$$\dot{V} = r\left(x - \bar{x}\right)\left(1 - \frac{C_0}{\overline{C_0}}\right) + r\left(x - \bar{x}\right)\left[\frac{C_0}{\overline{C_0}}\frac{\bar{x}}{k} - \frac{x}{k}\right] - \delta_1\left(C_0 - \overline{C_0}\right)^2$$

$$+\left(C_E - \overline{C_E}\right)\left[\left(1 - \frac{C_0}{\overline{C_0}}\right)u\right] + \left(C_E - \overline{C_E}\right)\left[a_1\left(C_0 - \overline{C_0}\right) - C_E + \overline{C_E}\frac{C_0}{\overline{C_0}}\right]$$

$$= \lambda_0 u + \lambda_1\left(rx\left(1-\frac{x}{k}\right)-\alpha_1 C_0 x\right)$$

$$+\lambda_2\left(a_1 C_E - \delta_1 C_0 + \theta_1\right)$$
$$+\lambda_3\left(-hC_E + \alpha_2 C_0 + u(t)\right)$$

$$\dot{\lambda}_1 = \frac{-\partial H}{\partial x} = \lambda_1\left(r - \frac{2rx}{k} - \alpha_1 C_0\right)$$

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial C_0} = \lambda_1\left(-\alpha_1 x\right) + \lambda_2\left(-\delta_1\right) + \lambda_3\left(\alpha_2\right)$$

$$\dot{\lambda}_3 = \frac{-\partial H}{\partial C_E} = \lambda_2\left(a_1\right) + \lambda_3\left(-h\right)$$

$$\lambda_2 = -\frac{h}{a_1}\lambda_0 \qquad \lambda_1 \alpha_1 x = -\lambda_0\left(\alpha_2 - \frac{h}{a_1}\right)$$

$$\lambda_1 \alpha_1 x \left(\frac{rx}{k} \right) = 0$$

$$\frac{du}{dt} = \delta_1 \left(\exp(-t) + \mu \right) - \eta u \ ,$$